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THE ENTREPRENEURIAL ECONOMY I: CONTRACTING UNDER KNIGHTIAN UNCERTAINTY

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This is the first of a series of papers in which we address the questions How do capitalist systems generate their dynamism and Why a capitalist economy is inherently different from a centrally planned one. We believe that, in order to study these issues, we must thoroughly reconsider the role played by entrepreneurs and financiers in actual economies. We claim that this role has been grossly misrepresented in classical theories, mainly because of the way the "uncertainty" is modeled. We begin by proposing a new definition of "innovation". This differs from any other definition previously given in the literature and reveals at once the inadequacy of the current theoretical paradigm. In fact, it shows that (a) Knightian uncertainty rather than Risk plays a crucial role in capitalist economies; (b) that two groups of agents, entrepreneurs and financiers, play a special role in that they deal with Knightian uncertainty; (c) that a crucial difference between centrally planned and capitalist systems might reside in the latter's ability to deal with Knightian uncertainty. The final part of this paper focuses on the role of entrepreneurs and financiers as micro actors. There, we study the problem of two parties contracting in a situation of Knightian Uncertainty. This is widely recognized as a very difficult problem, which has not be solved despite several attempts. Here, we solve the problem in a special case, which is nonetheless sufficient for our purposes.

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1. INTRODUCTION

This is the first of a series of papers devoted to the study of the Entrepreneurial Economy. In essence, we aim at addressing the following questions *How do capitalist systems generate their dynamism* and *Why a capitalist economy is inherently different from a centrally planned one*. Our fundamental belief is that, in order to study these issues, we must study the mechanisms of entrepreneur-ship and innovation in capitalist economies: the role of entrepreneurs in seeing commercial possibilities for developing and adopting products that exploit new technologies; the role of entrepreneurs in conceiving and developing new products and methods; the role of financiers in identifying entrepreneurs to back and to advise; and the incentives and disincentives for entrepreneurship inside established corporations. This means studying both the entrepreneur as a micro actor and the entrepreneurial economy as an interactive system. We begin our study with this paper. The first five sections lay down the foundations for the entire work. In those, our main concern is to identify and properly formalize the main forces at play in a capitalist system. We begin by quickly reviewing the way classical theories have dealt with the problem of Uncertainty, and explain the reason why we find this treatment dissatisfactory. In fact, getting a bit ahead of ourselves, some of developments contained in this paper will lend support to the thesis that it is precisely the classical treatment of Uncertainty that has lead to a gross misrepresentation of the role played by entrepreneurs and financiers in actual economies. Next, having suggested that one of the main distinctive features of a capitalist system might be its ability of generating "innovation", we inquire into the problem of formalizing the concept of innovation. While several definitions of innovation have been proposed in the literature, we find none of those fully satisfactory. Thus, in Section 4, we provide our own definition. While the reader will be the ultimate judge, we venture to claim that ours is a rather original contribution. Not only does our definition dramatically differ from any other previously given in the literature, but also it reveals the inadequacy of the current theoretical paradigm. In fact, our definition at once reveals (in a formal sense) that (a) Knightian uncertainty (also referred to as Ambiguity) rather than Risk plays a crucial role in capitalist economies; (b) that two groups of agents, entrepreneurs and financiers, play a special role in that they deal with Knightian uncertainty; (c) that a crucial difference between centrally planned and capitalist systems might reside in the latter's ability to deal with Knightian uncertainty. The final part of this paper focuses on the role of entrepreneurs and financiers as micro actors. Having found that the very reason for the existence of these two actors is the presence of Knightian Uncertainty, we have to face the difficult problem of two parties contracting in a situation of Ambiguity. This is widely recognized as a very difficult problem, which has not be solved despite several attempts. Here, we solve the problem in a special case, which is nonetheless sufficient for our purposes. We find that the optimal contract takes an especially nice form. This allows not only for an easy comparison with the known results in the theory of risk-sharing, but also permits – at the least at level of interpretation – to disentangle the role played by Ambiguity from that played by Risk.

2. RISK, AMBIGUITY AND THE INADEQUACY OF THE CLASSICAL MODEL

According to the classical Arrow-Debreu model, a centrally planned economy can achieve the same level of efficiency as a decentralized one. This is in sharp contrast with historical reality: in our day, capitalist economies have displayed both a higher level of efficiency and much larger accumulation of capital. We believe that the reason for this discrepancy resides mainly in the role played by financiers in capitalist economies. In fact, this is precisely what makes capitalist economies look sensibly different from centrally planned economies. Yet, this is not recognized in the classical model. There, financiers' only role is to exploit arbitrage opportunities, which would eventually disappear anyway under the force of perfect competition. In a nutshell, financiers play a role only in that they guarantee – through the competition among them – that an equilibrium will be reached. As we shall see, this trivialization of the role played by financiers (as well as of that played by entrepreneurs) is a consequence of an implicit yet crucial assumption about the way "Uncertainty" is modeled.

Unquestionably, one of the most salient features of actual economies is the widespread presence of uncertainty. Some of the ingredients for being either a successful entrepreneur or a successful financier are certainly the ability of correctly predicting how the uncertainty will be resolved, the ability of getting ready for that resolution and, perhaps, the ability of minimizing the damage in case of mistaken forecast. At any given point in time, we observe some companies going bankrupt while others rise to stardom. A phenomenon which, to the very least, suggests that the way one deals with uncertainty is anything but an uncontroversial matter.

In the classical theory, following the Bayesian tradition, uncertainty is modeled as Risk. This is done as follows. There is a given list of contingencies which is assumed to be known to all agents, and each agent's uncertainty is described by a probability distribution over the set of those contingencies. While (ex ante) different agents might have different views (i.e. different probability distributions), the information conveyed by the market eventually leads them to entertain the same view: at an economy's equilibrium no two agents are willing to bet against each other about the uncertainty's resolution. Thus, in classical theories, there is nothing uncontroversial about the way one deals with uncertainty.

The contrast between this prediction of the theory and what happens in real life is striking. Actual economic agents usually disagree about the resolution of uncertainty, and even the assumption of a list of contingencies known to all agents as well as the assumption of each agent having a probability distribution over such contingencies seem hardly tenable. It is an old idea, dating back to at least F. Knight, that some – and, perhaps, the most relevant – economic decisions are made in circumstances where the information available is too coarse to make full sense of the surrounding environment, where things look too fuzzy for having a probability distribution over a set of relevant contingencies. In such situations, Risk Theory is simply of no use.

This distinction between Risk and Ambiguity was fully appreciated by F. Knight, who attempted to build a theory of entrepreneurship and profit based on it. We adhere to this view. In fact, our main claim is that, granted the presence of Ambiguity, the main differences between a decentralized economy and a centrally planned one could be explained by the different ways each type of economy deals with Ambiguity. Of course, many steps will be necessary to support our claim and, in fact, along the way, the role of many classical economic actors will have to undergo a thorough reconsideration. Here is a quick list.

1. **Entrepreneurs**: Entrepreneurs are one of the two important sides in our story. Their main function will be that of generating "innovations". In modeling this, we will heavily depart from previous attempts in the literature. In our theory, innovations emerge when entrepreneurs hold a view of the world which is fundamentally different from that held by the market. In the process of generating innovations, entrepreneurs will be a primary source of Knightian Uncertainty.

2. **Financiers**: Financiers are the other important side. It is up to financiers to deal with the Knightian Uncertainty generated by the entrepreneurs. Classical theories view a financier as someone who tries to exploit existing arbitrage opportunities. We depart from this tradition heavily. For us, the essence of being a financier is to try to exploit opportunities that are not yet in existence.

3. **Markets**: Classical view holds that capitalist economies function through markets only. We will reverse this viewpoint dramatically. In our view, the very essence of a capitalist economy reveals itself through bilateral transactions, involving entrepreneurs and financiers. These occur outside the market, and cannot be replicated in a centrally planned economy.

4. **Type of contracts**: As the "innovation" process is inherently associated to Knightian Uncertainty, contracts signed by entrepreneurs and financiers would be different from the usual contracts devised in the literature. In fact, so far the Contract Theory literature has been confined to the analysis of risky situations. Part of our work will be devoted to explain what kind of contract would emerge in situations of Knightian Uncertainty. In general, one should expect that the

structure of transactions emerging from our theory would be sensibly different from that associated to classical theories.

3. STATES OF THE WORLD. OBJECTIVE STATES VS SUBJECTIVE STATES

Just like in the classical model, a basic concept in our theory is that of state of the world. Following the Bayesian tradition, a state of the world is a complete specification of all the parameters defining an environment. For instance, a state of the world for the economy would consist of a specification of temperature, humidity, consumers' tastes, technological possibilities, detailed maps of all possible planets, etc. According to this view, the future is uncertain because it is not known in advance which state will obtain. In principle (but this is clearly beyond human capabilities), one might come up with the full list of all possible states, and classical theories postulate that each and every agent would be described by a probability measure on such a list.

We depart from this tradition in that we do not assume the existence of a list of all possible states which is known to all agents. We do so for several reasons. First, we believe that this assumption is too artificial. Second, a theory built on such an assumption would not be testable, not even in principle. Third, and more importantly, we believe that, by making such an assumption, we would lose sight of the actual role played by entrepreneurs and financiers in actual economies.

In our work, we take a point of view that we might define as objective. At each point in time, the amount of assets existing in the economy is observable. Each of these assets pays contingent on a certain number of states of the world. The union (taken over all the assets) of all these states is then objectively given, in the sense that is it derived from observables. We call this set the set of publicly known states of the world, and denote it by SP. We assume that each and every agent in the economy is aware of all the states contained in SP.

Of course, there is no reason why each agent in the economy, individually considered, be restricted to hold the same view as the market. In other words, while we assume that each agent be aware of the set SP, we are also open to the possibility that each agent might consider states that are not in SP. An example might clarify. Suppose that a certain mine is known to contain an amount of x tons of gold and an amount of y tons of silicon. Suppose also that there are only two states in SP, a and b, where

a = tomorrow's market price of gold is 10 per tons and that of silicon is 0 per ton

b = tomorrow's market price of gold is 20 per tons and that of silicon is 0 per ton

Finally, suppose that the (market) probability of each state is 1/2. Then, today's market value of the mine is $15x$. However, some agent might believe that the set of possible tomorrow's states might be larger than a and b, and that it might contain a state where the price of silicon might be of 20 per ton. If the probability that such an agent assigns to this new state is sufficiently high, then the mine's value for the agent might be higher than $15x$. Later, we will see how this can be reconciled with the equilibrium hypothesis. For the moment, it would suffice to say that we admit that each agent i has a subjective state space S_i of the form $S_i = SP \cup I_i$, where I_i is the list of contingencies in agent i's set of states that are not publicly known.

4. INNOVATION

The idea of "innovation" and the way we model it is central to our theory. Unquestionably, the ability of "innovating" is one of the most distinguishing features of capitalist economies. Innovations occur in the form of new consumption goods, new technological processes, new institutions, new forms of organizations in trading activities, etc. We abstract from the differences existing across different types of innovation, and focus on what is common among them. For us, an innovation is defined as follows

Definition 1. An innovation is a set of states of the world which are not publicly known along with an asset which pays contingent on those states.

An example will clarify momentarily. For now, we should like to point out that the word "asset" in the definition should be interpreted in a broad sense. That is, by asset we mean any activity capable of generating economic value.

In order to illustrate the definition, let us imagine an economy where historically only two types of cakes have been consumed: carrot cakes and coconut cakes. Each year, each individual consumer might be of one of two types: either he likes carrot cakes (consumers of type 1) or coconut cakes (consumers of type 2) but not both. The fraction of the population made of consumers of type 1 varies from year to year according to some known stochastic process. Summing up, in our economy there are two productive processes: one for producing carrot cakes and one for coconut cakes. There is a continuum of tomorrow's states, where each state gives the fraction of consumers of type 1. These states are understood by everyone in the economy. That is, $SP = [0, 1]$ and a point x in $[0, 1]$ means that the fraction of type 1 consumers is x. Moreover, there is a given probability distribution on $[0, 1]$, which is known to everyone in the economy.

Now, suppose that an especially creative individual, whom we call E, comes into the scene and (a) figures out a new productive process that produces banana cakes;

(b) believes that each consumer, whether of type 1 or 2, would switch to banana cakes with probability 1/3 if given the opportunity. What is happening here is that agent E has: (1) imagined a whole set of new states, those in which consumers might like banana cakes (in fact, the subjective state space for agent E is two-dimensional, while SP is one-dimensional); (2) imagined that a non-negligible probability mass might be allocated to the extra dimension conditional on the consumers being given the chance to consume banana cakes; (3) figured out a devise (the productive process) that makes the new states capable of generating economic value.

Hopefully, the example has convincingly demonstrated that the definition given above is the "right" definition, that is, it conveys the essential features which identify any innovation (the new states along with the new activity). Our way of modeling innovation differs dramatically from previous attempts in the literature. Differently from those attempts, our way makes it clear that the process of innovation is truly associated to the appearance of new and fundamentally different possibilities: from the viewpoint of the innovator, both the state space and the space of production possibilities have higher dimensionality.

5. INNOVATION AND KNIGHTIAN UNCERTAINTY

In our story, the innovators are the entrepreneurs. But what is going to happen once they come up with an innovation? In the economy above, how are consumers going to react if they are told that banana cakes will be available? We follow up on the idea expressed above that an innovation is associated to a new scenario, something that the economy as whole has never experienced before. It is then natural to regard such a situation as one of Knightian uncertainty: the information available is (except, possibly, for the entrepreneur) too coarse to form a probability distribution on the relevant contingencies.

There is definitely something novel in the way Knightian uncertainty appears in our model: its source is not some devise (Nature) outside the economic system; rather, it is some of the economic actors – the entrepreneurs – who are a primary source of Knightian uncertainty. In our theory, one of the forces generating the dynamics of capitalist economies will be precisely this cycle

Entrepreneurs create Knightian uncertainty $\longrightarrow$ Financiers deal with it $\longrightarrow$ the economy takes a new shape $\longrightarrow \ldots$

5.1. Consumers and Financiers

All we have said so far leads to the problem of how economic agents would make decisions when facing Knightian uncertainty. In our setting, the problem takes the following form. Consider an agent i who has a subjective state space S_i and a certain probability distribution P_i on it. Now, suppose that our agent is told about another state, s, that he had not thought of before. The problem is to describe how such an agent behaves with respect to the state space $\{S_i, s\}$ (S_i union s) given that (Knightian uncertainty) he cannot form assessments about the likelihood of s.

Decision theorists have developed several models to deal with this problem, all of which stipulate that the behavior of agents facing Knightian uncertainty is described not by a single probability but rather by a set of those ([11], [17], [7], [2]). Following this literature, we are going to assume that when agent i has to evaluate assets that pay contingent on the state space $\{S_i, s\}$,he will use all the probability distributions on $\{S_i, s\}$ which are compatible with (whose conditional is) P_i on S_i. Then, we are going to label economic agents according to the attitude they display toward the Ambiguity. Precisely,

1) There is a group of agents in the economy, called *consumers*, whose subjective state space coincides with the publicly known set of states and who are ambiguity averse, in the sense that they always evaluate their options according to the worst probability (worst case scenario = maxmin expected utility);

2) There is a group of agents in the economy, called *financiers*, whose subjective state space coincides with the publicly known set of states and who are less ambiguity averse than the consumers. Financiers evaluate asset f by means of the functional

$$V(f) = \alpha(f) \min_{P \in \mathcal{C}} \int u(f) dP + (1 - \alpha(f)) \max_{P \in \mathcal{C}} \int u(f) dP \qquad (1)$$

where $\mathcal{C}$ is a set of probability measures on the new state space faced by the financier, u is the financier's utility on consequences and $\alpha(f) \in [0, 1]$. Intuitively, the coefficient $\alpha(f)$ represents the degree of Ambiguity aversion of the financiers (see [7]), and this degree is allowed to vary with the asset (=entrepreneurial project) to be evaluated. We suppose that for at least one asset f, $\alpha(f) < 1$. Clearly, we allow for the special case where the financiers' attitude toward Ambiguity does not depend on the project to be evaluated. In such a case, projects are evaluated by using the functional

$$V(f) = \alpha \min_{P \in \mathcal{C}} \int u(f) dP + (1 - \alpha) \max_{P \in \mathcal{C}} \int u(f) dP$$

where we suppose that $\alpha < 1$.

We believe that this categorization captures the essential (functional) distinction between the concept of "consumer" and "financier": a (pure) consumer is someone who rejects the unknown, a financier is somebody that is willing to bet on it. One might argue that the assumption that the agent's subjective state space is SP is natural in the case of consumers but it is not so in the case of financiers. This is not problematic, however, because a financier's subjective state space bigger than SP

can be easily accommodated in our framework by suitably re-defining the function $\alpha(f)$, which represents the financier's Ambiguity aversion.

Before concluding this part, we would like to give the reader some insights of what lies ahead. In a sequel paper, we will make an assumption guaranteeing that (in a sense to be made precise) the mass of consumers is much larger than the mass of financiers. This will imply that the market as a whole is strongly ambiguity averse (approximately, maximin expected utility). It will then follows that entrepreneurs will be unable to sell their projects on the market..

5.2. Innovation and financiers

By our categorization above, financiers are those who deal with the entrepreneurs. In traditional risk theories, the role of financiers is to try to exploit arbitrage opportunities. Here, their role is more subtle. In a way, they still try to exploit arbitrage opportunities, but these are not yet in existence. All the more, the very existence of these opportunities is not recognized by the market as explained above. This has two important consequences: first, the interaction between financiers and entrepreneurs has to occur through bilateral negotiations; second, there is nothing that guarantees success for the financiers. Below, we will study some of the delicate – and entirely novel – issues associated to the problem of two parties contracting in a situation of Knightian uncertainty. For now, we stress that financiers are the channel through which innovations can be transformed from mere ideas into a source of economic growth.

6. CONTRACTING BETWEEN ENTREPRENEURS AND FINANCIERS: PRELIMINARY CONSIDERATIONS

The remainder of this paper is devoted to studying the problem of contracting between entrepreneurs and financiers. As explained above, this problem is fundamentally different from the ones studied in the Contract Theory literature since here contracting takes places in a situation of Ambiguity. The informal description of the problem is as follows. An entrepreneur E has a new idea. Not having enough wealth to finance his project, the entrepreneur seeks a financier to obtain the necessary funds. According to our formalization above, the new idea consists of a set of states I which are not publicly known along with an asset X that pays contingent on the states in I. The entrepreneur has a clear idea (or, at least, he believes so) of the probabilistic description of the problem, and communicates it to the financier in order to induce the latter to provide the necessary funds. In other words, the entrepreneur has a probability distribution over I, and communicates it to the financier. Faced with this description, the financier extends his view of the world to incorporate the new contingencies devised by the entrepreneur, and does so in the way described above. Then, he evaluates the project by using his attitude toward ambiguity, and decides whether or not he deems the project worth financing. If he does so, then the parties will agree on a contract that specifies an initial transfer (possibly zero) between them as well as a schedule that specifies transfers between them conditional on the realization of states in I.

Before we move on to the formal description of this problem, a few observations are in order. First, there is an obvious adverse selection problem here: unlike what we said in our informal description, there is no guarantee that, in general, the entrepreneur will faithfully reveal the outcome that occurs contingent on a state in I. That is, there is no *a priori* guarantee that the entrepreneur would have no incentives of misrepresenting the profitability of his project. As it is well known, however, faithful representation on the part of the entrepreneur can be guaranteed by the financier by offering contracts with suitable properties. We will show that all contracts that we are going to determine below have these properties. Second, we abstract from issues of observability and verifiability of the states in I. That is, we assume that all states in I are both observable and verifiable. While we believe that these issues are important in real life contracting, we prefer to keep our inquiry focused on the problem of contracting under Ambiguity without mixing it with issues of different nature. Finally, restricting to contracts that specify transfer only contingent on states in I without considering also the publicly known states in SP is justified only if we assume that there exists Arrow securities for each and every state in SP. Again, since we want to keep our inquiry focused on the problems stemming from the presence of Ambiguity, we are going to assumed that this is indeed the case. Notice, however, that this assumption is fully consistent with our definition of SP: in fact, given that definition, this is a tautology rather than a genuine assumption.

7. CONTRACTING BETWEEN ENTREPRENEURS AND FINANCIERS: THE FORMAL PROBLEM

From now until the end of the paper we are going to study the problem of contracting between entrepreneurs and financiers. In this section, we begin by giving the formal definition of a contract. Then, we formally describe both the entrepreneur's and the financier's preferences, in particular their attitudes toward Ambiguity. Finally, we state the problem of finding an optimal contract.

7.1. Definition of the contract

An entrepreneur E comes up with an innovation. According to our definition, this consists of a set of "new" states S_E along with a function $X : S_E \longrightarrow \mathbb{R}$ that specifies the gain/loss that will be realized if state $s \in S_E$ obtains. Since our entrepreneur will be fixed for the remainder of the paper, we suppress the subscript E, and write simply S in the place of S_E. We denote by Σ the σ-algebra on S generated by X. Thus, the entrepreneur's innovation is described by a pair $((S, \Sigma), X)$, where (S, Σ) is a measurable space and X is a random variable on (S, Σ). By Doob's measurability theorem (see [1, Theorem 4.41]) any measurable function g on (S, Σ) has the form $g = \zeta \circ X$, where ζ is a Borel-measurable function $\mathbb{R} \longrightarrow \mathbb{R}$. The Banach space of all bounded measurable functions on (S, Σ) (with $\|g\|_\infty = \sup_{s \in S} |g(s)|$) is denoted by $B(\Sigma)$ and the set of its positive elements by $B^+(\Sigma)$.

Definition 2. A contract between an entrepreneur and a financier is a pair (T, Y), where $T \geq 0$ and $Y \in B(\Sigma)$.

The definition formalizes the following scheme. The financier pays T to the entrepreneur and in exchange gets a claim on part the amount $X(s)$, which obtains when $s \in S$ realizes. This claim may consist of the all $X(s)$ or just a part of it. The amount that the entrepreneur gets when $s \in S$ realizes is denoted by $Y(s)$ (which may be zero). Formally, this description is equivalent to the scheme where the financiers acquires ownership of the project before the state realizes. He does so by paying T to the entrepreneur. Then, he obtains the amount $X(s)$ when $s \in S$ realizes, and transfer $Y(s)$ to the entrepreneur. Notice that this description includes as special cases the following:

(a) The financier simply buys the project, and has no further obligation toward the entrepreneur. This obtain for $Y(s) = 0$, for every $s \in S$;

(b) The entrepreneur retains ownership of the project, but commits to paying the amount $Z(s) = X(s) - Y(s)$ to the financier when $s \in S$ realizes. He does so in exchange for an up front (that is, before the uncertainty resolves) payment of T;

(c) The entrepreneur transfers part of the ownership to the financier in exchange for T, and the parties agree to a sharing rule that specifies that when $s \in S$ realizes the amount $Z(s) = X(s) - Y(s)$ goes to the financier and the amount $Y(s)$ goes to the entrepreneur.

In a static setting, the distinction between case (b) and case (c) is purely a matter of interpretation because the contract is formally the same. Differently, in case (a) one can actually talk of transfer of ownership. This is an important case, whose determination requires to characterizes all those circumstances (as functions of the project X and of the parties' preferences) that lead to optimal solutions with the feature $Y(s) = 0$, for every $s \in S$. We will address this problem in a future inquiry. At the moment, we are going to be interested mainly in determining the form of a general contract, and in understanding the role played by Ambiguity in this type of problems.

7.2. Description of the entrepreneur

As we discussed above, the entrepreneur has (in his subjective opinion) a clear probabilistic view of the "new" world S he has envisioned. This view is represented by a (countably additive)[2] probability measure μ on (S, Σ), which he uses to evaluate the possible contracts that he might sign. Formally, we assume that the entrepreneur evaluates his options by means of the Subjective Expected Utility criterion

$$\int_S u_E(Y) d\mu \quad , \quad Y \in B(\Sigma)$$

Here, $u_E : \mathbb{R} \longrightarrow \mathbb{R}$ is the entrepreneur's utility for (monetary) outcomes. Regarding u_E, we make the following assumption:

Assumption 1 The entrepreneur's utility function u_E satisfies the following properties:

1. $u_E(0) = 0$;

2. u_E is strictly increasing and strictly concave;

3. u_E is continuously differentiable;

4. u_E is bounded.

Thus, in particular, we assume that the entrepreneur is risk averse.

7.3. Description of the financier

Unlike the entrepreneur, the financier does not have a (unambiguous) probabilistic description of the new world S: for him, lots of uncertainty surrounds S. Following the literature in decision theory, the financier's perception of the uncertainty is revealed in the way he evaluates the possible contracts: while the entrepreneur uses a linear functional (the SEU functional = no ambiguity), the financier will use a non-linear functional. Formally, we assume that the financier's preferences over contracts are represented by a functional $V_F : B(\Sigma) \longrightarrow \mathbb{R}$, and that V_F is a non-additive functional. The literature in decision theory has identified and axiomatized several different types of non-additive behavior: among those, most popular in the applications are Maxmin Expected Utility [11], Choquet Expected Utility [17] and Variational Preferences [12] and [13], which are all of the type (1) (see [8]; we refer to the recent survey of [10] for more on this subject). Here, we are going to assume that the financier's preferences are of the Choquet Expected Utility type. Formally,

[2] In terms of preferences, our assumption about the countable additivity of the measure means that the entrepreneur's preferences satisfy the Axiom of Monotone Continuity [3].

Assumption 2 The financier's evaluates contract by means of the functional $V_F : B(\Sigma) \longrightarrow \mathbb{R}$ defined by

$$\int_S u_F(Y)d\nu \qquad , \qquad Y \in B(\Sigma)$$

where $u_F : \mathbb{R} \longrightarrow \mathbb{R}$ is the financier's utility for money, ν is a capacity on Σ and the integral is taken in the sense of Choquet. In addition, we assume that the financier is risk-neutral and we take u_F to be the identity on $\mathbb{R}$. For the ease of the reader not acquainted with this literature, we collect a few facts about Choquet integrals in Appendix A.

7.4. Knowledge and uncertainty

When the financier and the entrepreneur get together, the latter describes to the former his project along with his beliefs about the likelihood of the various states in S. We assume that the entrepreneur declares truthfully his beliefs μ, which are then a common knowledge among the parties. Given that the financier's assessments of the uncertainty surrounding the project are formed independently of the entrepreneur's description, this assumption does not seem particularly demanding. Also, mainly for reasons of comparison with other parts of the contracting literature, we assume that the uncertainty on S is diffused. Precisely,

Assumption 3 We assume that:

1. $\mu \circ X^{-1}$ is nonatomic;

2. ν is continuous capacity (see Appendix A).

7.5. The problem of finding the optimal contract

From now on, we are going to assume that the random variable X which describes the profitability of the project is a positive random variable, that is $X \in B^+(\Sigma)$. This is without loss of generality since it can always be obtained by suitably re-normalizing the parties utility functions. Let W_0^E denote the entrepreneur's initial wealth (possibly zero). By signing contract (T, Y), the entrepreneur's wealth as a function of the sate $s \in S$ that will realize is given by

$$W^E(s) = W_0^E + T - X(s) + Y(s)$$

which is, clearly, a measurable function on (S, Σ).

A necessary condition for the financier to offer the contract is that his evaluation of the random variable $X - Y$ (the amount that he gets, as a function of the state, if he signs the contract) be at least as high as the amount T that he has to pay up front to the entrepreneur. In fact, the financier's evaluation of $X - Y$ might have to be strictly higher than T since by funding the entrepreneur the financier gives up other investment opportunities, in particular those available on the asset market as described by SP. Thus, the financier's *participation constraints* is

$$\int_S (X - Y)d\nu \geq (1 + \rho)T$$

where $\rho \geq 0$. The problem of finding the optimal contract can then be split into two parts: first, we are going to look for the solution of the problem

$$\begin{aligned} & \sup_{Y \in B(\Sigma)} \int_S u_E(W_0^E + T - X(s) + Y(s))d\mu \\ s.t. \quad -W_0^E & \leq Y \leq X \\ \int_S (X - Y)d\nu & \geq (1 + \rho)T \end{aligned}$$

and then we look for the optimal T. The second constraint in the optimization problem expresses two conditions: the right-hand inequality states that, in each state of the world, the transfer from the financier to the entrepreneur does not exceed the profitability of the project; while the left-hand inequality states that if there is a transfer from the entrepreneur to the financier, this will not exceed the entrepreneur's initial wealth. In our inquiry, however, we are going to be mainly interested in the case where entrepreneurs do not have initial wealth (at least to be devoted to the running the project). Thus, we are going to be focusing on the problem

$$\begin{aligned} & \sup_{Y \in B(\Sigma)} \int_S u_E(W_0^E + T - X(s) + Y(s))d\mu \qquad (2) \\ s.t. \quad 0 & \leq Y \leq X \\ \int_S (X - Y)d\nu & \geq (1 + \rho)T \end{aligned}$$

7.6. Truthful revelation of the profitability of the project

When studying a problem of contracting in a situation of uncertainty, one typically adds one more constraint to the ones we considered above. This is a monotonicity constraint that, in our case, would stipulate that the payment from the financier to the entrepreneur is an increasing function of X, that is $Y = \Xi \circ X$ for some increasing function $\Xi : \mathbb{R} \longrightarrow \mathbb{R}$. This would guarantee that the entrepreneur does not downplay the profitability of the project. For the moment, we are going to ignore this problem altogether. The reason is the following: in our main theorem, *we are going to show that the monotonicity of Y is a feature that appears in all optimal contracts that we determine.*

8. CONTRACTING BETWEEN ENTREPRENEURS AND FINANCIERS: THE SOLUTION

In this section, we are going to show that the contracting problem (2) between the entrepreneur and the financier admits a solution. Moreover, we are going to show that this solution is increasing in X. Thus, even in the case that the project profitability depends on (state-contingent) unobserved actions taken by the entrepreneur, there would be neither adverse selection nor moral hazard problems with an optimal contract. Our solution obtains under an assumption which guarantees a certain consistency between the financier's and the entrepreneur's assessments of the uncertainty. The formal property is stated in the following definition, which is a mild extension of a concept introduced in Ghossoub [9] (for the definition of comonotonic functions, see App. A)

Definition 3. Let ν be a capacity on Σ, μ be a measure on Σ and X be a random variable on (S, Σ). We say that ν is (μ, X) vigilant if for any Y_1,$Y_2 \in B^+(\Sigma)$ such that

(i) Y_1 and Y_2 have the same distribution under μ; and

(ii) Y_2 and X are comonotonic

the following holds

$$\int (X - Y_2) d\nu \geq \int (X - Y_1) d\nu$$

In our contracting framework, to say that ν *is* (μ, X) *vigilant means that the financier considers the entrepreneur's description* (μ, X) *of the project sufficiently credible.* Note that this is a *subjective statement* on the part of the financier. In fact, one can depict the following the story. A entrepreneur envisions the "new" world S and comes up with his new idea (μ, X). Then, he goes to a financier to ask for funding, and tells him about the "new" world S and the project (μ, X). The financier forms his view of S, which is described by ν, and decides how credible the entrepreneur's project is. If he deems it sufficiently credible, then they would start negotiating. If not, the entrepreneur would take leave and seek for a financier with a different opinion. Thus, the appearance of assumptions of the vigilance-type should not be surprising, as ultimately these are conditions for both parties to believe in the mutual profitability of the project. An interesting problem would be to determine the minimal level of credibility required for a certain contract to be signed or, inversely, what are the contracts that the parties are willing to sign for a given credibility level. We leave this for future research. Before proceeding, however, we should like to stress that in the special case where the capacity ν is a measure, the assumption of vigilance is a weakening of the monotone likelihood ratio property frequently assumed in the contracting literature to deal with problems stemming from the asymmetry in the information. We refer the reader to Ghossoub [9], for the relation between the two properties in a context of Risk. We can now state our main result.

THEOREM 1. *If ν is (μ, X) vigilant, then Problem (2) admits a solution Y which is comonotonic with X.*

Thus, as we have already stressed, our result implies that there no incentive for the entrepreneur to misrepresent the profitability of his project. The proof of the Theorem is in Appendix D. In the next sections, we will relate our result to a classical result of Arrow [3] and Borch [5], and we will say more about the optimal contract.

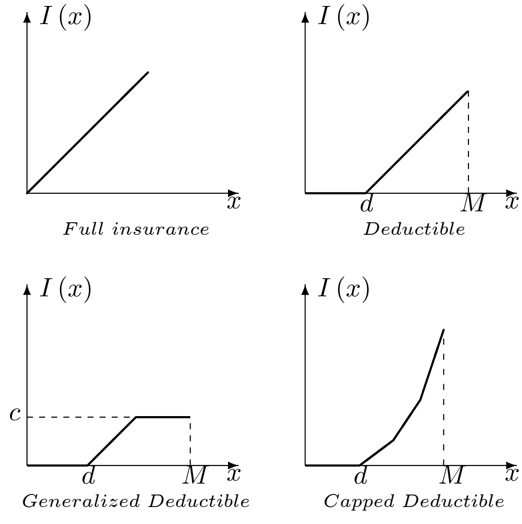
9. AN ASIDE: THE INSURANCE MODEL (ARROW AND BORCH)

9.1. The classical model (Arrow and Borch [3] and [5])

If in Problem (2) we assume that (a) in addition to the entrepreneur, the financier also evaluates the possible contracts by means of an Expected Utility criterion; and (b) the financier and the entrepreneur use the *same* probability measure, then we obtain a problem of the same form as the classical insurance problem studied by Arrow [3] and Borch [5]. Precisely, the classical insurance problem is the following

$$
\begin{aligned}
&\sup_{Y \in B(\Sigma)} \int_S u(W_0^E - \Pi - X(s) + Y(s))\, d\mu \\
s.t.\quad 0 \;&\leq\; Y \leq X \\
(1+\rho)\int_S Y\, d\nu \;&\leq\; \Pi
\end{aligned}
$$

We recall the following categorization of contracts from the insurance literature (where we have set $Y = I \circ X$)



In our setting, the full-insurance contract corresponds to a pure loan contract, while the other three correspond to situations where the entrepreneur transfers (in full or in part) ownership to the financier in exchange for an up front payment and a profit-sharing scheme (see Subsection 7.1, above). Arrow-Borch classical result is the following

THEOREM 2 (Arrow-Borch [3] and [5]; see also Raviv [16]). *There exists a deductible contract that is optimal for the insurance problem.*

We stress that this is a pure Risk-sharing result: the two parties fully agree about the description of the uncertainty, which they both reduce to risk in the exact same way. The only reason leading the parties to signing a contract is the different shape of the utility for money (the entrepreneur is risk-averse while the financier is risk-neutral).

9.2. Ghossoub [9]

Recently, Ghossoub [9] obtained a remarkable improvement over the Arrow-Borch result. Ghossoub's setting is still a Risk-setting in that both parties evaluate the contracts by means of an expected utility criterion. Ghossoub, however, allows the parties to entertain different views of the risk involved as he allows for the two probability measures to be different. Lack a common prior, reconciles the assumption of common knowledge (see above) with Aumann's Agreement Theorem [4]. Ghossoub shows that in this situation the optimal contract is a generalized deductible. Notably, he achieves this results by using rather novel techniques. Our main result here is based on those. We refer the reader to the Appendices B and C and to Ghossoub [9] for more detail.

10. INTERPRETATION OF THE SOLUTION

In this subsection, we are going to show that the optimal contract takes the form of a generalized deductible if the financier is sufficiently Ambiguity loving (see the observation following the Proposition below). Then, we are going to interpret the result by taking as a benchmark the linear deductible contract of Arrow-Borch.

PROPOSITION 1. *Suppose that the financier evaluates contracts by using a sub-modular capacity. Then, there exists an optimal contract which is a generalized deductible.*

The assumption that the financier evaluates contracts by using a sub-modular capacity is equivalent, by a result of Schmeidler [17], to the assumption that the financier's functional is of the form

$$\max_{z \in \mathcal{C}} \int \Psi dZ \qquad , \qquad \Psi \in B(\Sigma)$$

where $\mathcal{C}$ is a compact, convex set of probability measures on (S, Σ); equivalently, the coefficient $\alpha(f)$ in the functional (1) is identically equal to 0, thus making the financier Ambiguity-loving. The proof of the Proposition is in Appendix E. By virtue of this proposition, we see that (in the case $\alpha(f) \equiv 0$) the difference between the optimal contract in purely risky-situations and the optimal contract in situations of Ambiguity consists in the non-linearity of the profit-sharing schedule. This is due to the fact that the parties have different views about the uncertainty surrounding the project: concavity parts in this schedule might account for the fact that the entrepreneur is more optimistic about certain outcomes than the financiers (that is, considers those outcomes more likely than the financier's (non-additive) assessment), with the situation being reversed in the parts of convexity.

APPENDICES

A. Choquet integral

Definition 4. A *capacity* on a measurable space $(S, \mathcal{G})$ is a set function $\nu : \mathcal{G} \longrightarrow [0,1]$such that

(1) $\nu(\varnothing) = 0$;

(2) $\nu(S) = 1$; and

(3) $A, B \in \mathcal{G}$ *and* $A \subset B \Longrightarrow \nu(A) \leq v(B)$.

Definition 5. A capacity on $(S, \mathcal{G})$ is continuous from above (resp. below) if for any sequence $\{A_n\} \subset \mathcal{G}$ such that $A_{n+1} \subset A_n$ (resp. $A_{n+1} \supset A_n$) for each n, it holds that

$$\lim_{n\to\infty} \nu(A_n) = \nu\left(\bigcap_{n=1}^{\infty} A_n\right) \qquad (resp.\ \lim_{n\to\infty} \nu(A_n) = \nu\left(\bigcup_{n=1}^{\infty} A_n\right))$$

A capacity that is continuous both from above and below is said to be continuous.

Remark 1. Probability distortions are examples of continuous capacities.

Definition 6. Given a capacity ν and a function $\psi \in B(\mathcal{G})$, the Choquet integral of ψ wrt ν is defined by

$$\int \psi d\nu = \int_0^{\infty} \nu(\{s \in S : \psi(s) \geq t\})dt + \int_{-\infty}^{0} [\nu(\{s \in S : \psi(s) \geq t\}) - 1]dt$$

where the integrals on the RHS are taken in the sense of Riemann.

Unlike the Lebesgue integral, the Choquet integral is not additive. One of its characterizing properties, however, is that it respects additivity on comonotonic functions.

Definition 7. Two functions $Y_1, Y_2 \in B(\mathcal{G})$ are comonotonic if for all $s, s' \in S$

$$[Y_1(s) - Y_1(s')][Y_2(s) - Y_2(s')] \geq 0$$

As noticed, if $Y_1, Y_2 \in B(\mathcal{G})$ are comonotonic then

$$\int (Y_1 + Y_2)d\nu = \int Y_1 d\nu + \int Y_2 d\nu$$

For more on the Choquet integral, we refer to [14].

B. A useful Lemma

The following result is a special case of Helly's First Theorem. Its proof can be found in [6, Lemma 13.15].

LEMMA 1. *If* $\{f_n\}_n$ *is a uniformly bounded sequence of nondecreasing real-valued functions on some closed interval* $\mathcal{I}$ *in* $\mathbb{R}$ *with bound* N *(i.e.* $|f_n(x)| \leq N, \forall x \in \mathcal{I}.\forall n \geq 1$*), then there exists a nondecreasing real-valued bounded function* f^* *on* $\mathcal{I}$*, also with bound* N*, and a subsequence of* $\{f_n\}_n$ *that converges pointwise to* f^* *on* $\mathcal{I}$*.*

C. Rearrangements and supermodularity

The results in Appendix are from Ghossoub [9] to which we refer the reader for proofs and additional results.

C.1The Nondecreasing Rearrangement

Let $(S, \mathcal{G}, P)$ be a probability space, and let $X \in B^+(\mathcal{G})$ be a continuous random variable (i.e., $P \circ X^{-1}$ is non-atomic) with range $X(S) = [0, M]$.Let Σ be the σ-algebra generated by X, and let

$$\phi(B) = P(\{s \in S : X(s) \in B\}) = P \circ X^{-1}(B)$$

for any Borel subset B of $\mathbb{R}$.

Let $I : [0, M] \to [0, M]$ be any Borel-measurable map. Then there exists a ϕ-a.s. unique nondecreasing Borel-measurable map $\tilde{I} : [0, M] \to [0, M]$ which is ϕ-equimeasurable with I, in the sense that for any $\alpha \in [0, M]$,

$$\phi(\{t \in [0, M] : I(t) \leq \alpha) = \phi(\{t \in [0, M] : \tilde{I}(t) \leq \alpha)$$

$\tilde{I}$ is called the *nondecreasing* ϕ*-rearrangement of* I. Now, define $Y = I \circ X$ and $\tilde{Y} = \tilde{I} \circ X$. Since both I and $\tilde{I}$ are Borel-measurable mapping of $[0, M]$ into itself, it follows that $Y, \tilde{Y} \in B^+(\Sigma)$. Note also that $\tilde{Y}$ is non-decreasing in X, in the sense that if $s_1, s_2 \in S$ are such that $X(s_1) \leq X(s_2)$ then $\tilde{Y}(s_1) \leq \tilde{Y}(s_2)$ and that Y and $\tilde{Y}$ are equimeasurable, that is for any $\alpha \in [0, M]$, $P(\{s \in S : Y(s) \leq \alpha) = P(\{s \in S : \tilde{Y}(s) \leq \alpha)$.

We will call $\tilde{Y}$ a *nondecreasing* P*-rearrangement of* Y *with respect to* X, and we shall denote it by $\tilde{Y}_P$. Note that $\tilde{Y}_P$ is P-a.s. unique. Note also that if Y_1 and Y_2 are P-equimeasurable and if $Y_1 \in \mathcal{L}_1(S, \mathcal{G}, P)$, then $Y_2 \in \mathcal{L}_1(S, \mathcal{G}, P)$ and $\int \psi(Y_1)dP = \int \psi(Y_2)dP$ for any measurable function ψ such that the integral exists.

C.2 Supermodularity and Hardy-Littlewood Inequalities

A *partially ordered set* (*poset*) is a pair $(A, \succsim)$ where $\succsim$ is a reflexive, transitive and antisymmetric binary relation on A. For any $x, y \in A$ we denote by $x \vee y$ (resp. $x \wedge y$) the least upper bound (resp. greatest lower bound) of the set $\{x, y\}$. A poset $(A, \succsim)$ is a *lattice* when $x \vee y, x \wedge y \in A$ for every $x, y \in A$. For instance, the Euclidian space $\mathbb{R}^n$ is a lattice for the partial order $\succcurlyeq$ defined as follows: for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $y = (y_1, ... y_n) \in \mathbb{R}^n$, we write $x \succcurlyeq y$ when $x_i \geq y_i$, for each $i = 1, \ldots, n$. It is then easy to see that $x \vee y = (\max(\mathrm{x}_1, y_1), \ldots, \max(\mathrm{x}_n, y_n))$ and $x \wedge y = (\min(x_1, y_1), \ldots, \min(x_n, y_n))$.

Definition 8. Let $(A, \succsim)$ be a lattice. A function $L : A \longrightarrow \mathbb{R}$ is supermodular if for each $x, y \in A$

$$L(x \vee y) + L(x \wedge y) \geq L(x) + L(y)$$

In particular, a function $L : \mathbb{R}^2 \to \mathbb{R}$ is *supermodular* if for any $x_1, x_2, y_1, y_2 \in \mathbb{R}$ with $x_1 \leq x_2$ and $y_1 \leq y_2$, we have

$$L(x_2, y_2) + L(x_1, y_1) \geq L(x_1, y_2) + L(x_2, y_1)$$

It is easily seen that the supermodularity of a function $L : \mathbb{R}^2 \to \mathbb{R}$ is equivalent to the function $\eta(y) = L(x + h, y) - L(x, y)$ being nondecresing for any $x \in \mathbb{R}$ and $h \geq 0$.

Example 1. The following are useful examples of supermodular functions:

(1) If $g : \mathbb{R} \to \mathbb{R}$ is concave, and $a \in \mathbb{R}$, then the function $L_1 : \mathbb{R}^2 \to \mathbb{R}$ defined by $L_1(x, y) = g(a - x + y)$

(2) If $f : \mathbb{R} \to \mathbb{R}$ is concave, and $a \in \mathbb{R}$, then the function $L_2 : \mathbb{R}^2 \to \mathbb{R}$ defined by $L_2(x, y) = f(a - x + y)$

(3)The function $L_3 : \mathbb{R}^2 \to \mathbb{R}$ defined by $L_3(x, y) = -(y - x)^+$ is supermodular.

Lemma 2. *Let $(S, \mathcal{G}, P)$ be a probability space, let $X \in B^+(\mathcal{G})$ be a continuous random variable and let $Y \in B^+(\Sigma)$. Denote by $\tilde{Y}_P$ the nondecreasing P-rearrangement of Y with respect to X. Then,*

(a) $0 \leq Y \leq X$ implies $0 \leq \tilde{Y}_P \leq X$;

(b) If L is a supermodular, $P \circ X^{-1}$-integrable function on the range of X, then

$$\int L(X, Y) dP \leq \int L(X, \tilde{Y}_P) dP$$

D. Proof of the main theorem

In order to prove Theorem 1, we begin with a couple of Lemmata. Let us denoted by $\mathcal{F}_{SB}$ the feasibility set for Problem 2

$$\mathcal{F}_{SB} = \left\{ Y \in B(\Sigma) : 0 \leq Y \leq X \text{ and } \int (X - Y) d\nu \geq (1 + \rho)T = H \right\}$$

and assume that $\mathcal{F}_{SB}$ is nonempty. Let $\mathcal{F}_{SB}^{\uparrow}$ be the set of all the $Y \in \mathcal{F}_{SB}$ which, in addition, are comonotonic with X

$$\mathcal{F}_{SB}^{\uparrow} = \{Y = I \circ X \in \mathcal{F}_{SB} : I \text{ is nondecreasing}\}$$

LEMMA 3. *If ν is (μ, X)-vigilant, then $\mathcal{F}_{SB}^{\uparrow} \neq \varnothing$.*

Proof. By assumption, $\mathcal{F}_{SB} \neq \varnothing$. Choose any $Y = I \circ X \in \mathcal{F}_{SB}$, and let $\tilde{Y}_\mu$ denote the nondecreasing μ-rearrangement of Y with respect to X. Then (i) $\tilde{Y}_\mu = \tilde{I} \circ X$ where $\tilde{I}$ is nondecreasing, and (ii) $0 \leq \tilde{Y}_\mu \leq X$, by Lemma 2. Furthermore, since ν is (μ, X)-vigilant, it follows that $\int (X - \tilde{Y}_\mu) d\nu \geq \int (X - Y) d\nu$. But $\int (X - Y) d\nu \geq H$ as $Y \in \mathcal{F}_{SB}$. Hence, $\tilde{Y}_\mu \in \mathcal{F}_{SB}^{\uparrow}$ and $\mathcal{F}_{SB}^{\uparrow} \neq \varnothing$. ∎

DEFINITION 9. If $Y_1, Y_2 \in \mathcal{F}_{SB}$, we say that Y_2 is a Pareto improvement over Y_1 if the following hold

$$\begin{aligned} (1) \int u_E(W_0^E + F - X + Y_2)\, d\mu \quad &\geq \quad \int u_E(W_0^E + F - X + Y_1)\, d\mu \\ (2) \int (X - Y_2) d\nu \quad &\geq \quad \int (X - Y_1) d\nu \end{aligned}$$

LEMMA 4. *Suppose that ν is (μ, X)-vigilant. If $Y \in \mathcal{F}_{SB}$, then there is some $Y^* \in \mathcal{F}_{SB}^{\uparrow}$ which is a Pareto-improvement over Y.*

Proof. By the previous lemma, $\mathcal{F}_{SB}^{\uparrow} \neq \varnothing$. Choose any $Y \in \mathcal{F}_{SB}$, and let $Y^* := \tilde{Y}_\mu$, where $\tilde{Y}_\mu$ denotes the nondecreasing μ-rearrangement of Y with respect to X. Then $Y^* \in \mathcal{F}_{SB}^{\uparrow}$, as in the proof of Lemma 4. Moreover, since the utility function u_E is concave, the function

$$\mathcal{U}(x, y) \equiv u_E(W_0^E + T - x + y)$$

is supermodular. Thus, it follows from Lemma 2 that

$$\int u_E(W_0^E + F - X + Y^*) d\mu \geq \int u_E(W_0^E + F - X + Y) d\mu$$

Moreover, since ν is (μ, X)-vigilant

$$\int (X - Y^*) d\nu \geq \int (X - Y) d\nu$$

Thus, $Y^* \in \mathcal{F}_{SB}^{\uparrow}$ is a Pareto-improvement over $Y \in \mathcal{F}_{SB}$. ∎

Proof of Theorem 1. By Lemma 4, we can choose a maximizing sequence $\{Y_n\}_n$ in $\mathcal{F}_{SB}^{\uparrow}$ for Problem 2. That is,

$$\lim_{n\to+\infty}\int u_E(W_0^E+T-X+Y_n)d\mu = N \equiv \sup_{Y\in B^+(\Sigma)}\left\{u_E(W_0^E+T-X+Y_n)d\mu\right\} < +\infty$$

Since $0 \leq Y_n \leq X \leq M := \|X\|_\infty$, the sequence $\{Y_n\}_n$ is uniformly bounded. Moreover, for each $n \geq 1$ we have $Y_n = I_n \circ X$, with $I_n : [0,M] \to [0,M]$. Consequently, the sequence $\{I_n\}_n$ is a uniformly bounded sequence of nondecreasing Borel-measurable functions. Thus, by Lemma 1, there is a nondecreasing function $I^* : [0,M] \to [0,M]$ and a subsequence $\{I_m\}_m$ of $\{I_n\}_n$ such that $\{I_m\}_m$ converges pointwise on $[0,M]$ to I^*. Hence, I^* is also Borel-measurable, and so $Y^* := I^* \circ X \in B^+(\Sigma)$ is such that $0 \leq Y^* \leq X$. Moreover, the sequence $\{Y_m\}_m$, $Y_m = I_m \circ X$, converges pointwise to Y^*. Thus, $\{X - Y_m\}_m$ is uniformly bounded and converges pointwise to $\{X - Y^*\}$. By the Assumption that ν is continuous, it follows from a Dominated Convergence-type Theorem [15, Theorem 7.16][3] that

$$H \leq \lim_{m\to+\infty}\int (X - Y_m)d\nu = \int (X - Y^*)d\nu$$

and so $Y^* \in \mathcal{F}_{SB}^{\uparrow}$. Now, by continuity and boundedness of the function u, and by Lebesgue's Dominated Convergence Theorem [1, Theorem 11.21], we have

$$\begin{aligned}\int u(W_0^E+T-X+Y^*)d\mu &= \lim_{m\to+\infty}\int u(W_0^E+T-X+Y_m)d\mu \\ &= \lim_{n\to+\infty}\int u(W_0^E+T-X+Y_n)d\mu = N\end{aligned}$$

Hence Y^* solves Problem 2. ∎

Proof of Proposition 1

Proof of Proposition 1. The optimal contract problem is

$$\begin{aligned}& \sup_{Y\in B(\Sigma)}\int_S u_E(W_0^E+T-X(s)+Y(s))d\mu \qquad (3)\\ s.t.\quad 0 &\leq Y \leq X \\ \max_{Z\in\mathcal{C}}\int_S (X-Y)dZ &= \int_S (X-Y)d\nu \geq (1+\rho)T = H\end{aligned}$$

[3] The theorem of Pap [15] is for the *Šipoš integral*, or the *symmetric Choquet integral*. However, the latter coincides with the Choquet integral for nonnegative functions, as per Pap [15], Theorem 7.9 on p. 153.

where $\mathcal{C} = anticore(\nu)$. Consider the family of contracting problems indexed by $Z \in \mathcal{C}$ and defined by

$$\sup_{Y \in B(\Sigma)} \int_S u_E(W_0^E + T - X(s) + Y(s))d\mu \tag{4}$$

$$\begin{aligned} s.t. \quad 0 &\leq Y \leq X \\ \int_S (X - Y)dZ &\geq H \end{aligned}$$

By [9], each of these problems admits a solution Y_Z which is a generalized deductible. Now, let $Z' \in \mathcal{C}$ and let Y^* denote a solution of problem (3). Since $Y_{Z'}$ solves the problem defined by Z', we have

$$H \leq \int_S (X - Y_{Z'})dZ \leq \max_{Z \in \mathcal{C}} \int_S (X - Y_{Z'})dZ$$

and $Y_{Z'}$ is feasible in problem (3). By the assumption that Y^* solves problem (3), we must have

$$\int_S u_E(W_0^E + T - X(s) + Y^*(s))d\mu \geq \int_S u_E(W_0^E + T - X(s) + Y_{Z'}(s))d\mu \tag{5}$$

Now, either we can find a Z' such that (5) is an equality, and the claim is proved, or

$$\int_S u_E(W_0^E + T - X(s) + Y^*(s))d\mu > \int_S u_E(W_0^E + T - X(s) + Y_{Z'}(s))d\mu$$

for every $Z' \in \mathcal{C}$. Since $Y_{Z'}$ solves the problem defined by Z', this implies that Y^* must not be feasible in any the problems (4), that is

$$\int_S (X - Y^*)dZ' < H$$

for every $Z' \in \mathcal{C}$. But, again by the assumption that Y^* solves problem (3), we then have

$$\int_S (X - Y^*)dZ' < H \leq \max_{Z \in \mathcal{C}} \int_S (X - Y^*)dZ$$

for every $Z' \in \mathcal{C}$, which (since $(X - Y^*) \in B(\Sigma)$) contradicts the fact that $\mathcal{C}$ is compact and convex. ∎

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