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Entrepreneurship, Ambiguity, and the Shape of Innovation Contracts

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ENTREPRENEURSHIP, AMBIGUITY, AND THE SHAPE OF INNOVATION CONTRACTS

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Abstract. In Amarante, Ghossoub, and Phelps (AGP) [2], we proposed a model of innovation and entrepreneurship where the entrepreneur generates innovation, innovation generates Ambiguity for all economic agents except the entrepreneur, and the financier deals with this Ambiguity through bilateral contracts that we called *innovation contracts*. Under a requirement on the financier's ambiguous beliefs, we showed the existence and monotonicity of optimal innovation contracts. Moreover, when the financier is ambiguity-loving in the sense of Schemeidler [26], we showed that the problem of contracting for innovation under Ambiguity can be reduced to a situation of non-ambiguous but heterogeneous Bayesian beliefs. This is important since the latter situations have been examined by Ghossoub [10, 12], and the solutions can be characterized in that case. In this paper, we consider a special case of the setting of AGP [2] which will allow us to fully characterize an optimal innovation contract, all the while maintaining a situation where the financier has ambiguous beliefs.

1. Introduction, Preliminary Definitions, and Setup

In a fixed time horizon, any financial instrument in a financial market can be seen as an asset whose monetary value at any given point in time is contingent on the realizations of some prevailing uncertainty. In the Bayesian decision-theoretic tradition, uncertainty is typically represented by a space of *states of the world*, also called a *state space*. Financial instruments can then be seen as functions from the space into the real line, where a real number represents the monetary value of this financial instrument in a given state of the world. At any given point in time, the collection of assets, or financial instruments existing in the economy is observable by all economic agents, at least in principle. Each asset is a function of a set of contingencies. The union taken over all assets of these contingencies is what we call the *set of publicly observable states*.

Hence, a given collection F of observable financial instruments will generate a collection OS of *publicly observable* sates of the world that all economic agents will agree upon. All economic agents

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observe the space OS, but in addition, each economic agent might envisage states that are not in OS. These are *subjective sates*. Hence, each agent i has a subjective state space SS^i of the form

$$SS^i = OS \cup S^i,$$

where S^i is a list of sates of the world envisaged by agent i, but are not publicly known (i.e., $OS \cap S^i = \varnothing$).

1.1. **Entrepreneurs and Innovation.** The way in which the idea of innovation is defined in Amarante, Ghossoub, and Phelps (AGP) [2] is general. Innovation is defined roughly as any envisaged financial instrument that pays contingent on the presently observable states of the world in OS, but also pays contingent on some presently unobservable states of the world that are envisaged by some innovator, also called the entrepreneur in AGP [2].

Definition 1.1. An *innovation* is a set of states of the world which are not publicly observable, along with an asset which pays contingent on those states and on the observable ones.

Here, the word asset should be interpreted broadly as an economic activity capable of generating value, and is measured in monetary terms. Hence, an innovation is a pair $(S \cup OS, f)$ such that $S \cup OS$ is a newly envisaged state space, and $f : S \cup OS \to \mathbb{R}$ is a monetary measurement of the economic value of some newly conceived asset. The object f can be seen as a new financial instrument. The process of innovation not only creates new financial instruments, but also foresees new states of the world.

Definition 1.2. An *entrepreneur* is any economic agent who generates an innovation.

Consequently, an entrepreneur e can be described by a pair (SS^e, X_e). Entrepreneurs are the innovators, and their entrepreneurial endeavours enrich the economy through newly conceived assets and newly envisaged future contingencies. To a large extent, the entrepreneurial activity in an economy inherently generates the dynamism of that economy.

1.2. **Financiers and Ambiguity.** Consider an economy with an observable sate space OS constructed as discussed above, and let $\mathcal{A}$ denote the collection of all economic agents in this economy. Suppose that an economic agent $e \in \mathcal{A}$ is an entrepreneur, described by a pair (SS^e, X_e), where $SS^e = OS \cup S^e$, S^e is a collection of non-observable states envisaged by e, and $X_e : SS^e \to \mathbb{R}$ is the monetary measurement of agent e's innovation. We may assume that e is Bayesian on the state space SS^E, having a probability measure P^e on $(SS^e, \mathcal{G}^e)$ representing his beliefs, where $\mathcal{G}^e$ is a σ-algebra of subsets of SS^e, called events. By the very definition of S^e, any other economic agent $a \in \mathcal{A} \backslash \{e\}$ will have no *a priori* knowledge of S^e, and hence of SS^e. We may then assume that the agent a is non-Bayesian, having ambiguous beliefs over the space $(SS^e, \mathcal{G}^e)$. In other words, the information available to agent a is neither accurate enough, nor complete enough for him to be able to formulate an additive Bayesian prior belief. Therefore, by its nature, any entrepreneurial activity generates ambiguity in the economy, in the sense just described.

When facing ambiguity generated by the entrepreneurial activity of some entrepreneur $e \in \mathcal{A}$, the rest of the economic agents in the economy can be broadly divided into two groups:

- a collection $\mathcal{C} \subset \mathcal{A}$ of agents that are (strictly) averse to ambiguity. These agents are called *consumers* by AGP [2];
- a collection $\mathcal{F} \subset \mathcal{A}$ of agents that are either ambiguity-neutral or ambiguity-seeking. These agents are called *financiers* by AGP [2].

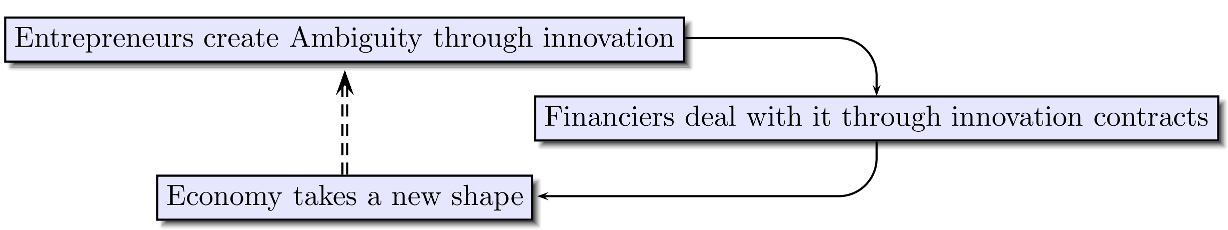
Entrepreneurship in an economy not only generates dynamism by foreseeing new contingencies and new financial instruments, but also inherently generates ambiguity. This, in turn, classifies the economic agents into three categories: the entrepreneur himself, the consumers who are ambiguity-averse, and the financiers who are not ambiguity-averse. Modern decision theory, also called Neo-Bayesian decision theory, has developed many models of choice under uncertainty to accommodate for the presence of ambiguity and ambiguity-aversion. We refer to the recent survey of Gilboa and Marinacci [14, 15] for more on this topic. In this paper, the particular model of decision under ambiguity that we use is that of the *Choquet Expected-Utility* (CEU) model of Schmeidler [26]. In the CEU model, ambiguity is represented by a non-additive probability (also called a capacity) on the state space (see Appendix A.1).

The information available to the entrepreneur is the information generated by the financial instrument $X_e : S^e \cup OS \to \mathbb{R}$. That is, the information available to e is the σ-algebra $\Sigma_e := \sigma\{X_e\}$ of subsets of $S^e \cup OS$ generated by X^e. Without loss of generality, we assume that the random variable X is nonnegative (see AGP [2]). Let $B(\Sigma_e)$ denote the space of all bounded, Σ_e-measurable functions from $S^e \cup OS$ into $\mathbb{R}$, and let $B^+(\Sigma_e)$ denote the cone of nonnegative elements of $B(\Sigma_e)$. By a classical result [1, Th. 4.41], the elements of $B^+(\Sigma_e)$ are the functions of the form $I \circ X_e$, where $I : X_e(S^e \cup OS) \to \mathbb{R}^+$ is a bounded, Borel-measurable function. We can then assume that both the entrepreneur and the financier have preferences over the elements of $B^+(\Sigma_e)$, since these are precisely the "innovation contracts" that both parties wish to examine.

Notation. *Henceforth, the measurable space $(S^e \cup OS, \Sigma_e)$ will be denoted by (S, Σ), for convenience of notation, and the subscripts and superscripts " e" will be dropped all throughout.*

2. Ambiguity and Innovation Contracts

As in AGP [2], the interaction between entrepreneurs and financiers is central to our study of innovation. The role of the financier is as essential to the dynamism of an economy as is the role of an entrepreneur. If the entrepreneur is the mother of innovation, the financier is the midwife. Without a financier, an entrepreneur might not be able to give shape to his innovation. It is this interaction between entrepreneurs and financiers that generates dynamism in the economy. This interaction boils down to a problem of contracting between an entrepreneur and a financier that AGP [2] calls a problem of *contracting for innovation.* In essence, entrepreneurs create innovations, innovations generate Ambiguity, and financiers deal with this Ambiguity through bilateral contracts for innovation:



2.1. **Setting.** The entrepreneur e seeks financing from a financier φ to cover the costs of finalizing his innovation. The entrepreneur gives a description of his innovation to the financier, including a description of the envisaged new states of the world and the new financial instrument that will serve as a monetary measurement of this innovation. Although the entrepreneur is assumed to be Bayesian, having an additive probabilistic assessment of the unobservable sates of the world, and although these sates are communicated to the financier, there is no *a priori* reason why the financier will also behave as a Bayesian decision maker on the unobservable sates. The financier will be assumed to behave according to CEU, having a non-additive probabilistic assessment of the unobservable sates.

The contracting situation that might occur between the entrepreneur and the financier can be described as follows. For a given initial lump-sum financing H that the financier gives to the entrepreneur, the latter promises to transfer the total monetary value X of the innovation to the former, and to receive a monetary compensation $I(X)$ contingent on the monetary amount of the innovation. The problem that will ensue is to determine an optimal monetary transfer rule $I(X)$. This will be clearer once the formal setting is introduced.

As mentioned above, the entrepreneur will inform the financier about S, X, and hence also Σ. The entrepreneur will be assumed to be Bayesian on (S, Σ), whereas the financier will be assumed to have non-additive ambiguous beliefs on (S, Σ). In other words, the entrepreneur's preferences $\succcurlyeq_e$ over $B^+(\Sigma)$ have a Subjective Expected-Utility (SEU) representation, yielding a utility function $u_e : \mathbb{R} \rightarrow \mathbb{R}$ for monetary outcomes, and a (unique countably additive[1]) probability measure P on (S, Σ). That is, for each $Y, Z \in B^+(\Sigma)$,

$$Y \succcurlyeq_e Z \iff \int u_e \circ Y \, dP \geqslant \int u_e \circ Z \, dP$$

We will also make the following assumption.

Assumption 2.1. *X is a continuous random variable on the probability space (S, Σ, P). That is, the probability measure P is such that the image measure $P \circ X^{-1}$ is nonatomic[2].*

Moreover, the utility function u is bounded and satisfies Inada's [17] conditions. Specifically,

(1) *u is bounded;*
(2) *$u(0) = 0$;*
(3) *u is strictly increasing and strictly concave;*
(4) *u is continuously differentiable; and,*
(5) *$u'(0) = +\infty$ and $\lim_{x \to +\infty} u'(x) = 0$.*

In particular, the entrepreneur is assumed to be Bayesian and risk-averse[3]. The financier $\varphi \in \mathcal{F}$, on the other hand, has an ambiguous assessment of the situation. We will assume that the financier's preferences $\succcurlyeq_\varphi$ over the elements of $B^+(\Sigma)$ have a Choquet-Expected Utility (CEU) representation as in Schmeidler [26], yielding a utility function $u_\varphi : \mathbb{R} \rightarrow \mathbb{R}$ for monetary outcomes, and a capacity υ on (S, Σ). See Appendix A.1 for a brief description of the ideas of a capacity and a Choquet integral with respect to a capacity. Therefore, for each $Y, Z \in B^+(\Sigma)$,

$$Y \succcurlyeq_\varphi Z \iff \int u_\varphi \circ Y \, d\upsilon \geqslant \int u_\varphi \circ Z \, d\upsilon$$

where integration is in the sense of Choquet (Definition A.3). We will also make the following assumption.

[1] Countable additivity can be obtained by assuming that preferences satisfy the Arrow-Villegas *Monotone Continuity* axiom [5].

[2] A finite measure η on a measurable space $(\Omega, \mathcal{G})$ is said to be *nonatomic* if for any $A \in \mathcal{G}$ with $\eta(A) > 0$, there is some $B \in \mathcal{G}$ such that $B \subsetneq A$ and $0 < \eta(B) < \eta(A)$.

[3] In expected-utility theory, risk-aversion is equivalent to the concavity of the utility function. This is not necessarily true for non-expected-utility preferences.

Assumption 2.2.

(1) *The capacity υ is continuous (Definition A.2)*

(2) *The utility function u_{φ} is linear.*

If the financier were Bayesian on the state space (S, Σ), with preferences having a SEU representation, then the assumption of linearity of his utility function is tantamount to an assumption of risk-neutrality, which is a standard assumption in the literature on contracting and related problems. We will maintain the linearity assumption here. Note, also, that since a utility function is given up to positive affine transformations [26, pp. 578-579], we can then assume without loss of generality that the utility function v is simply the identity function, so that for each $Y, Z \in B^{+}(\Sigma)$,

$$Y \succcurlyeq_{\varphi} Z \iff \int Y \, d\upsilon \geqslant \int Z \, d\upsilon$$

2.2. **Contracting for Innovation under Ambiguity.** Formally, the innovation contract is a pair $(H, Y) \in \mathbb{R}^{+} \times B^{+}(\Sigma)$, where $H \geqslant 0$ is the initial lump-sum payment that the financier gives to the entrepreneur in exchange of the transfer of the total monetary value X of the innovation; and $Y = I \circ X \in B^{+}(\Sigma)$ is a repayment schedule from the financier to the entrepreneur, which the financier will promise to commit to. A repayment form the financier to the entrepreneur will always be a nonnegative amount, and it will never exceed the total monetary value of the innovation itself. In other words, a proper repayment schedule $Y \in B^{+}(\Sigma)$ will satisfy $Y \leqslant X$.

The entrepreneur has initial wealth W_0^e (which can be zero), and after entering into an innovation contract with the financier, his wealth in the state of the world $s \in S$ is given by

$$W(s) = W_0^e + H - X(s) + Y(s)$$

After an initial investment of H, the financier will receive $X(s) - Y(s)$, in each state $s \in S$, and the formal problem of contracting for innovation the AGP [2] considers is the following.

$$\begin{aligned} &\sup_{Y \in B(\Sigma)} \int u_e \left(W_0^e + H - X + Y\right) dP \\ &s.t. \quad 0 \leqslant Y \leqslant X \\ &\int (X - Y) \, d\upsilon \geqslant (1 + \rho) H \end{aligned} \tag{2.1}$$

where $\rho \geqslant 0$ is called a *loading factor*. Problem (2.1) has been studied by AGP [2], and we refer to the latter for a discussion of problem (2.1), including a description of the constraints involved. For the sake of completeness, we will review here some of their results. The first result sates that when the capacity υ satisfies a property called *vigilance* – initially introduced by Ghossoub [10] – there exists an optimal repayment scheme $Y^* = I^* \circ X$ which is comonotonic with X (Definition A.4), i.e., such that the function I^* is nondecreasing. This is an important result because such contracts imply a truthful revelation of the realizations of X.

Definition 2.3. The capacity υ is said to be *vigilant* if for any $Y, Z \in B^{+}(\Sigma)$ that satisfy

(1) Y and Z are identically distributed for P (i.e., $P \circ Y^{-1} = P \circ Z^{-1}$), and

(2) Y is comonotonic with X,

it follows that $\int (X - Y)\ dv \geqslant \int (X - Z)\ dv$.

Theorem 2.4 (AGP [2]). *If the capacity v is vigilant, then there exists an optimal solution Y^* to problem (2.1), and Y^* is comonotonic with X.*

The second result of AGP [2] that we will review here states that if the financier's non-additive belief v is submodular (i.e., concave – Definition A.5), then the problem could be reduced to a problem where no ambiguity exists. Specifically, consider the following family of problems, indexed by a probability measure μ on (S, Σ).

$$
\begin{aligned}
&\text{For a given probability measure } \mu \text{ on } (S,\Sigma)\,, \\
&\sup_{Y \in B(\Sigma)} \int u_e \left(W_0^e + H - X + Y \right) dP \\
&s.t. \quad 0 \leqslant Y \leqslant X \\
&\int (X - Y)\, d\mu \geqslant (1 + \rho)\, H
\end{aligned}
\tag{2.2}
$$

It is well-known that in the CEU model, concavity of the capacity v indicates an attitude of ambiguity-seeking. This was initially discussed in Schmeidler [26]. In light of our previous discussion of consumers and financiers, it seems natural that the financier, who is by definition not ambiguity-averse, is such that v is a concave capacity. Moreover, a classical result [26, pp. 583-584] sates that when v is concave, there exists a nonempty weak*-compact and convex collection of probability measures $\mathcal{AC}_v$ (called the *anti-core* of v) such that for each $Y \in B^+(\Sigma)$,

$$
\int Y\ dv = \max_{\mu \in \mathcal{AC}_v} \int Y\ d\mu
$$

Corollary 2.5 (AGP [2]). *If v is a concave capacity with anti-core $\mathcal{AC}_v$, and if each μ in $\mathcal{AC}_v$ is vigilant, then there exists a $\mu^* \in \mathcal{AC}_v$ such that a solution to problem (2.2) with measure μ^* is comonotonic with X and is a solution to problem (2.1) as well.*

This result is important mainly because it reduces the initial problem form a situation of ambiguity to a situation of non-ambiguous, but heterogeneous beliefs. The latter class of problems has been investigated by Ghossoub [10, 12].

3. A Full Characterization of Innovation Contracts in a Special Case

Here we consider a special case of the model of contracting for innovation introduced by AGP [2], which will allow us to fully characterize the shape of an optimal contract. This full characterization is helpful in practice since it permits to actually compute the optimal innovation contract as a function of the underlying innovation. However, this requires some additional assumptions.

Namely, we suppose first that $v = T \circ Q$, for some probability measure Q on (S, Σ) and some function $T : [0, 1] \to [0, 1]$, increasing, concave and continuous, with $T(0) = 0$ and $T(1) = 1$. Then $T \circ Q$ is a continuous submodular capacity on (S, Σ). Then the entrepreneur's problem becomes the following.

$$\begin{aligned} & \sup_{Y \in B(\Sigma)} \int u_e \left(W_0^e + H - X + Y\right) dP \\ & s.t. \quad 0 \leqslant Y \leqslant X \\ & \qquad \int (X - Y) \ dT \circ Q \geqslant (1+\rho) H \end{aligned} \tag{3.1}$$

Based on the results of Gilboa [13], we may assume that the distortion function T and the probability measure Q are *subjective*, i.e., they are determined entirely from the financier's preferences, since v is[4]. We will also assume that X is a continuous random variable on the probability space (S, Σ, Q). Specifically:

Assumption 3.1. *We assume that $v = T \circ Q$, where:*

(1) *Q is a probability measure on (S, Σ) such that $Q \circ X^{-1}$ is nonatomic;*
(2) *$T : [0,1] \to [0,1]$ is increasing, concave and continuously differentiable; and,*
(3) *$T(0) = 0$, $T(1) = 1$, and $T'(0) < +\infty$.*

We will also assume that the lump-sum start-up financing H that the entrepreneur receives from the financier guarantees a nonnegative wealth process for the entrepreneur. Specifically, we shall assume the following.

Assumption 3.2. *$X \leqslant W_0^e + H$, P-a.s.*

For each $Z \in B^+(\Sigma)$, let $F_Z(t) = Q\big(\{s \in S : Z(s) \leqslant t\}\big)$ denote the distribution function of Z with respect to the probability measure Q, and let $F_X(t) = Q\big(\{s \in S : X(s) \leqslant t\}\big)$ denote the distribution function of X with respect to the probability measure Q. Let $F_Z^{-1}(t)$ be the left-continuous inverse of the distribution function F_Z (that is, the *quantile function* of Z), defined by

$$F_Z^{-1}(t) = \inf \Big\{ z \in \mathbb{R}^+ : F_Z(z) \geqslant t \Big\}, \ \forall t \in [0,1] \tag{3.2}$$

Definition 3.3. Denote by $\mathcal{A}Quant$ the collection of all quantile functions f of the form F^{-1}, where F is the distribution function of some $Z \in B^+(\Sigma)$ such that $0 \leqslant Z \leqslant X$.

That is, $\mathcal{A}Quant$ is the collection of all quantile functions f that satisfy the following properties:

(1) $f(z) \leqslant F_X^{-1}(z)$, for each $0 < z < 1$;
(2) $f(z) \geqslant 0$, for each $0 < z < 1$.

Denoting by $Quant = \Big\{ f : (0,1) \to \mathbb{R} \ \Big| \ f \text{ is nondecreasing and left-continuous} \Big\}$ the collection of all quantile functions, we can then write $\mathcal{A}Quant$ as follows:

$$\mathcal{A}Quant = \Big\{ f \in Quant : 0 \leqslant f(z) \leqslant F_X^{-1}(z), \ \text{for each } 0 < z < 1 \Big\} \tag{3.3}$$

[4] [13, Th. 3.1] also yields that both T and P are unique.

By Lebesgue's Decomposition Theorem [1, Th. 10.61] there exists a unique pair (P_{ac}, P_s) of (non-negative) finite measures on (S, Σ) such that $P = P_{ac} + P_s$, $P_{ac} << Q$, and $P_s \perp Q$. That is, for all $B \in \Sigma$ with $Q(B) = 0$, we have $P_{ac}(B) = 0$, and there is some $A \in \Sigma$ such that $Q(S \backslash A) = P_s(A) = 0$. It then also follows that $P_{ac}(S \backslash A) = 0$ and $Q(A) = 1$. Note also that for all $Z \in B(\Sigma)$, $\int Z \, dP = \int_A Z \, dP_{ac} + \int_{S \backslash A} Z \, dP_s$. Furthermore, by the Radon-Nikodým Theorem [6, Th. 4.2.2] there exists a Q-a.s. unique Σ-measurable and Q-integrable function $h : S \to [0, +\infty)$ such that $P_{ac}(C) = \int_C h \, dQ$, for all $C \in \Sigma$. Consequently, for all $Z \in B(\Sigma)$, $\int Z \, dP = \int_A Zh \, dQ + \int_{S \backslash A} Z \, dP_s$. Moreover, since $P_{ac}(S \backslash A) = 0$, it follows that $\int_{S \backslash A} Z \, dP_s = \int_{S \backslash A} Z \, dP$. Thus, for all $Z \in B(\Sigma)$, $\int Z \, dP = \int_A Zh \, dQ + \int_{S \backslash A} Z \, dP$.

Moreover, since $h : S \to [0, +\infty)$ is Σ-measurable and Q-integrable, there exists a Borel-measurable and $Q \circ X^{-1}$-integrable map $\phi : X(S) \to [0, +\infty)$ such that $h = dP_{ac}/dQ = \phi \circ X$. We will also make the following assumption.

Assumption 3.4. *The Σ-measurable function $h = \phi \circ X = dP_{ac}/dQ$ is anti-comonotonic with X, i.e., ϕ is nonincreasing.*

Since $Q \circ X^{-1}$ is nonatomic (by Assumption 3.1), it follows that $F_X(X)$ has a uniform distribution over $(0, 1)$ [9, Lemma A.21], that is, $Q\big(\{s \in S : F_X(X)(s) \leqslant t\}\big) = t$ for each $t \in (0, 1)$. Letting $U := F_X(X)$, it follows that U is a random variable on the probability space (S, Σ, Q) with a uniform distribution on $(0, 1)$. Consider the following quantile problem:

$$
\begin{aligned}
&\text{For a given } \beta \geqslant (1+\rho) H, \\
&\qquad \sup_f \; V(f) = \int u_e\big(W_0^e + H - f(U)\big) \; \phi\left(F_X^{-1}(U)\right) \; dQ \\
&\qquad s.t. \quad f \in \mathcal{A}Quant \\
&\qquad\qquad \int T'(1-U) \, f(U) \; dQ = \beta
\end{aligned}
\tag{3.4}
$$

The following theorem characterizes the solution of problem (3.1) in terms of the solution of the relatively easier quantile problem given in problem (3.4), provided the previous assumptions hold. The proof is given in Appendix B.

Theorem 3.5. *Under the previous assumptions, there exists a parameter $\beta^* \geqslant (1+\rho) H$ such that if f^* is optimal for problem (3.4) with parameter β^*, then the function*

$$Y^* = \big(X - f^*(U)\big) \mathbf{1}_A + X \mathbf{1}_{S \backslash A}$$

is optimal for problem (3.1).

In particular, $Y^* = X - f^*(U)$, Q-a.s. That is, the set E of states of the world s such that $Y^*(s) \neq \Big(X - f^*(U)\Big)(s)$ has probability 0 under the probability measure Q (and hence $v(E) = T \circ Q(E) = 0$). The contract that is optimal for the entrepreneur will be seen by the financier to be almost surely equal to the function $X - f^*(U)$.

Another immediate implication of Theorem 3.5 is that the states of the world to which the financier assigns a zero "probability" are sates where the innovation contract is a full transfer rule. On the set of all other states of the world, the innovation contract deviates from a full transfer rule by the function $f^*(U)$.

Under the following two assumptions, it is possible to fully characterize the shape of an optimal innovation contract. This is done in Corollary 3.8.

Assumption 3.6. *The Σ-measurable function $h = \phi \circ X = dP_{ac}/dQ$ is such that ϕ is left-continuous.*

Assumption 3.7. *the function $t \mapsto \frac{T'(1-t)}{\phi\left(F_X^{-1}(t)\right)}$, defined on $t \in (0,1) \setminus \{t : \phi \circ F_x^{-1}(t) = 0\}$, is nondecreasing.*

Conditions similar to Assumption 3.7 have been used in several recent studies dealing with some problem of demand under Ambiguity, where the latter is introduced into the study via a distortion of probabilities. For instance,

- In studying portfolio choice under prospect theory [20, 27], Jin and Zhou [19] impose a similar monotonicity assumption [19, Assumption 4.1] to that used in our Assumption 3.7;
- To characterize the solution to a portfolio choice problem under Yaari's [28] dual theory of choice, He and Zhou [16] impose a monotonicity assumption [16, Assumption 3.5] which is also similar to our Assumption 3.7;
- In studying the ideas of greed and leverage within a portfolio choice problem under prospect theory, Jin and Zhou [18] use an assumption [18, Assumption 2.3] which is similar to our Assumption 3.7;
- Carlier and Dana [4] study an abstract problem of demand for contingent claims. When the decision maker's (DM) preferences admit a Rank-Dependent Expected Utility representation [23, 24], Carlier and Dana [4] show that a similar monotonicity condition to that used in our Assumption 3.7 is essential to derive some important properties of solutions to their demand problem [4, Prop. 4.1, Prop. 4.4]. Also, when the DM's preferences have a prospect theory representation, then Carlier and Dana [4] impose a monotonicity assumption [4, eq. (5.8)] similar to our Assumption 3.7.

When the previous assumptions hold, we can give an explicit characterization of an optimal contract, as follows.

Corollary 3.8. *Under the previous assumptions, there exists a parameter $\beta^* \geqslant (1+\rho) H$ such that an optimal solution Y^* for problem (3.1) takes the following form:*

$$Y^* = \left(X - \max\left[0, \min\left\{F_X^{-1}(U), f^*_{\lambda^*}(U)\right\}\right]\right)\mathbf{1}_A + X\mathbf{1}_{S\setminus A}$$

where for each $t \in (0,1) \setminus \{t : \phi \circ F_x^{-1}(t) = 0\}$,

$$f^*_{\lambda^*}(t) = W_0^e + H - \left(u_e'\right)^{-1}\left(\frac{-\lambda^* T'(1-t)}{\phi\left(F_X^{-1}(t)\right)}\right)$$

and λ^* is chosen so that

$$\int_0^1 T'(1-t)\max\left[0, \min\left\{F_X^{-1}(t), f^*_{\lambda^*}(t)\right\}\right] dt = \beta^*$$

The proof of Corollary 3.8 is given in Appendix C. Note that if Assumption 3.4 holds, then Assumption 3.6 is a weak assumption. Indeed, any monotone function is Borel-measurable, and hence "almost contiunous", in view of Lusin's Theorem [8, Theorem 7.5.2]. Also, any monotone function is almost surely continuous, for Lebesgue measure.

Appendix A. Background Material

A.1. **Capacities and the Choquet Integral.** Here, we summarize the basic definitions about capacities, Choquet integrals and Šipoš integrals. The proofs of the statements listed below can be found, for instance, in [21] or [22].

Definition A.1. A (normalized) *capacity* on a measurable space (S, Σ) is a set function $\upsilon : \Sigma \to [0,1]$ such that

(1) $\upsilon(\varnothing) = 0$;
(2) $\upsilon(S) = 1$; and
(3) $A, B \in \Sigma$ *and* $A \subset B \Longrightarrow \upsilon(A) \leqslant \upsilon(B)$.

Definition A.2. A capacity υ on (S, Σ) is continuous from above (resp. below) if for any sequence $\{A_n\}_{n \geqslant 1} \subseteq \Sigma$ such that $A_{n+1} \subseteq A_n$ (resp. $A_{n+1} \supseteq A_n$) for each n, it holds that

$$\lim_{n \to +\infty} \upsilon(A_n) = \upsilon\left(\bigcap_{n=1}^{+\infty} A_n\right) \quad \left(\text{resp. } \lim_{n \to +\infty} \upsilon(A_n) = \upsilon\left(\bigcup_{n=1}^{+\infty} A_n\right)\right)$$

A capacity that is continuous both from above and below is said to be continuous.

Definition A.3. Given a capacity υ and a function $\psi \in B(\Sigma)$, the Choquet integral of ψ w.r.t. υ is defined by

$$\int \phi \, d\upsilon = \int_0^{+\infty} \upsilon(\{s \in S : \phi(s) \geqslant t\}) \, dt + \int_{-\infty}^0 [\upsilon(\{s \in S : \phi(s) \geqslant t\}) - 1] \, dt$$

where the integrals on the RHS are taken in the sense of Riemann.

Unlike the Lebesgue integral, the Choquet integral is not additive. One of its characterizing properties, however, is that it respects additivity on comonotonic functions.

Definition A.4. Two functions $Y_1, Y_2 \in B(\Sigma)$ are comonotonic if for all $s, s' \in S$

$$\Big[Y_1(s) - Y_1(s')\Big]\Big[Y_2(s) - Y_2(s')\Big] \geqslant 0$$

As mentioned above, if $Y_1, Y_2 \in B(\Sigma)$ are comonotonic then

$$\int (Y_1 + Y_2) \, d\upsilon = \int Y_1 d\upsilon + \int Y_2 d\upsilon$$

Definition A.5. A capacity υ on (S, Σ) is submodular (or concave) if for any $A, B \in \Sigma$

$$\upsilon(A \cup B) + \upsilon(A \cap B) \leqslant \upsilon(A) + \upsilon(B)$$

It is supermodular (or convex) if the reverse inequality holds for any $A, B \in \Sigma$.

As a functional on $B(\Sigma)$, the Choquet integral $\int \cdot \, d\upsilon$ is concave (resp. convex) if and only if υ is submodular (resp. supermodular).

Proposition A.6. *Let υ be a capacity on (S, Σ).*

1. *If $Y \in B(\Sigma)$ and $c \in \mathbb{R}$, then $\int (Y + c) \; d\upsilon = \int Y \; d\upsilon + c$.*
2. *If $A \in \Sigma$ then $\int \mathbf{1}_A \; d\upsilon = \upsilon(A)$.*
3. *If $Y \in B(\Sigma)$ and $a \geqslant 0$, then $\int a \; Y \; d\upsilon = a \; \int Y \; d\upsilon$.*
4. *If $Y_1, Y_2 \in B(\Sigma)$ are such that $Y_1 \leqslant Y_2$, then $\int Y_1 \; d\upsilon \leqslant \int Y_2 \; d\upsilon$.*
5. *If υ is submodluar, then for any $Y_1, Y_2 \in B(\Sigma)$, $\int (Y_1 + Y_2) \; d\upsilon \leqslant \int Y_1 \; d\upsilon + \int Y_2 \; d\upsilon$.*

Definition A.7. The *Šipoš integral*, or the *symmetric Choquet integral* (see [22]), is a functional $\check{S} : B(\Sigma) \to \mathbb{R}$ defined by

$$\check{S}(Y) = \int Y^+ d\upsilon - \int Y^- d\upsilon$$

where the integrals on the RHS are taken in the sense of Choquet and Y^+ (resp. Y^-) denotes the positive (resp. negative) part of $Y \in B(\Sigma)$. Obviously, the Šipoš integral coincides with the Choquet integral for positive functions.

A.2. **Nondecreasing Rearrangements.**

All the definitions and results that appear in this Appendix are taken from Ghossoub [10, 11, 12] and the references therein. We refer the interested reader to Ghossoub [10, 11, 12] for proofs and for additional results.

A.2.1. *The Nondecreasing Rearrangement.* Let $(S, \mathcal{G}, P)$ be a probability space, and let $X \in B^+(\mathcal{G})$ be a continuous random variable (i.e., $P \circ X^{-1}$ is a nonatomic Borel probability measure) with range $X(S) = [0, M]$. Denote by Σ the σ-algebra generated by X, and let

$$\phi(B) := P\Big(\{s \in S : X(s) \in B\}\Big) = P \circ X^{-1}(B),$$

for any Borel subset B of $\mathbb{R}$.

For any Borel-measurable map $I : [0, M] \to \mathbb{R}$, define the *distribution function* of I as the map $\phi_I : \mathbb{R} \to [0, 1]$ given by $\phi_I(t) := \phi\big(\{x \in [0, M] : I(x) \leqslant t\}\big)$. Then ϕ_I is a nondecreasing right-continuous function.

Definition A.8. Let $I : [0, M] \to [0, M]$ be any Borel-measurable map, and define the function $\widetilde{I} : [0, M] \to \mathbb{R}$ by

$$\widetilde{I}(t) := \inf \Big\{ z \in \mathbb{R}^+ : \phi_I(z) \geqslant \phi\big([0, t]\big) \Big\} \tag{A.1}$$

Then $\widetilde{I}$ is a nondecreasing and Borel-measurable mapping of $[0, M]$ into $[0, M]$ such that I and $\widetilde{I}$ are ϕ-equimeasurable, in the sense that for any $\alpha \in [0, M]$,

$$\phi\Big(\{t \in [0, M] : I(t) \leqslant \alpha\}\Big) = \phi\Big(\{t \in [0, M] : \widetilde{I}(t) \leqslant \alpha\}\Big)$$

Moreover, if $\overline{I} : [0, M] \to \mathbb{R}^+$ is another nondecreasing, Borel-measurable map which is ϕ-equimeasurable with I, then $\overline{I} = \widetilde{I}$, ϕ-a.s. $\widetilde{I}$ is called the nondecreasing ϕ-rearrangement of I.

Now, define $Y := I \circ X$ and $\widetilde{Y} := \widetilde{I} \circ X$. Since both I and $\widetilde{I}$ are Borel-measurable mappings of $[0, M]$ into itself, it follows that $Y, \widetilde{Y} \in B^+(\Sigma)$. Note also that $\widetilde{Y}$ is nondecreasing in X, in the sense that if $s_1, s_2 \in S$ are such that $X(s_1) \leqslant X(s_2)$ then $\widetilde{Y}(s_1) \leqslant \widetilde{Y}(s_2)$, and that Y and $\widetilde{Y}$ are P-equimeasurable. That is, for any $\alpha \in [0, M]$, $P\Big(\{s \in S : Y(s) \leqslant \alpha\}\Big) = P\Big(\{s \in S : \widetilde{Y}(s) \leqslant \alpha\}\Big)$.

We will call $\widetilde{Y}$ a **nondecreasing P-rearrangement of Y with respect to X**, and we shall denote it by $\widetilde{Y}_P$. Note that $\widetilde{Y}_P$ is P-a.s. unique. Note also that if Y_1 and Y_2 are P-equimeasurable and if $Y_1 \in L_1(S, \mathcal{G}, P)$, then $Y_2 \in L_1(S, \mathcal{G}, P)$ and $\int \psi(Y_1)\ dP = \int \psi(Y_2)\ dP$, for any measurable function ψ such that the integrals exist.

A.2.2. *Supermodularity and Hardy-Littlewood Inequalities.* A *partially ordered set* (*poset*) is a pair $(A, \gtrsim)$, where $\gtrsim$ is a reflexive, transitive and antisymmetric binary relation on A. For any $x, y \in A$, we denote by $x \vee y$ (resp. $x \wedge y$) the least upper bound (resp. greatest lower bound) of the set $\{x, y\}$. A poset $(A, \gtrsim)$ is a *lattice* when $x \vee y, x \wedge y \in A$ for every $x, y \in A$. For instance, the Euclidian space $\mathbb{R}^n$ is a lattice for the partial order $\gtrsim$ defined as follows: for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, we write $x \gtrsim y$ when $x_i \geqslant y_i$, for each $i = 1, \ldots, n$.

Definition A.9. Let $(A, \gtrsim)$ be a lattice. A function $L : A \to \mathbb{R}$ is said to be *supermodular* if for each $x, y \in A$,

$$L(x \vee y) + L(x \wedge y) \geqslant L(x) + L(y)$$

In particular, a function $L : \mathbb{R}^2 \to \mathbb{R}$ is *supermodular* if for any $x_1, x_2, y_1, y_2 \in \mathbb{R}$ with $x_1 \leqslant x_2$ and $y_1 \leqslant y_2$, we have

$$L(x_2, y_2) + L(x_1, y_1) \geqslant L(x_1, y_2) + L(x_2, y_1)$$

It is then easily seen that supermodularity of a function $L : \mathbb{R}^2 \to \mathbb{R}$ is is equivalent to the function $\eta(y) = L(x + h, y) - L(x, y)$ being nondecreasing for any $x \in \mathbb{R}$ and $h \geqslant 0$.

Example A.10. The following are useful examples of supermodular functions on $\mathbb{R}^2$:

(1) If $g : \mathbb{R} \to \mathbb{R}$ is concave and $a \in \mathbb{R}$, then the function $L_1 : \mathbb{R}^2 \to \mathbb{R}$ defined by $L_1(x, y) = g(a - x + y)$ is supermodular;

(2) The function $L_2 : \mathbb{R}^2 \to \mathbb{R}$ defined by $L_2(x, y) = -(y - x)^+$ is supermodular;

(3) If $\eta : \mathbb{R} \to \mathbb{R}^+$ is a nonincreasing function, $h : \mathbb{R} \to \mathbb{R}$ is concave and nondecreasing, and $a \in \mathbb{R}$, then the function $L_3 : \mathbb{R}^2 \to \mathbb{R}$ defined by $L_3(x, y) = h(a - y)\ \eta(x)$ is supermodular.

Lemma A.11. *Let $Y \in B^{+}(\Sigma)$, and denote by $\widetilde{Y}_P$ the nondecreasing P-rearrangement of Y with respect to X. Then,*

(1) *If L is a supermodular and $P \circ X^{-1}$-integrable function on the range of X then:*

$$\int L\Big(X, Y\Big)\ dP \leqslant \int L\left(X, \widetilde{Y}_P\right)\ dP$$

(2) *If $0 \leqslant Y \leqslant X$ then $0 \leqslant \widetilde{Y}_P \leqslant X$.*

Appendix B. Proof of Theorem 3.5

B.1. **"Splitting".** Recall that by Lebesgue's Decomposition Theorem [1, Th. 10.61] there exists a unique pair (P_{ac}, P_s) of (nonnegative) finite measures on (S, Σ) such that $P = P_{ac} + P_s$, $P_{ac} << Q$, and $P_s \perp Q$. That is, for all $B \in \Sigma$ with $Q(B) = 0$, we have $P_{ac}(B) = 0$, and there is some $A \in \Sigma$ such that $Q(S \backslash A) = P_s(A) = 0$. It then also follows that $P_{ac}(S \backslash A) = 0$ and $Q(A) = 1$. In the following, the Σ-measurable set A on which Q is concentrated is assumed to be fixed all throughout. Consider now the following two problems:

$$\begin{aligned}
&\text{For a given } \beta \geqslant (1+\rho) H, \\
&\sup_{Y \in B(\Sigma)} \int_A u_e\big(W_0^e + H - X + Y\big)\ dP \\
&s.t. \quad 0 \leqslant Y \leqslant X \\
&\int (X - Y)\ dT \circ Q = \beta
\end{aligned} \tag{B.1}$$

and

$$\begin{aligned}
&\sup_{Y \in B(\Sigma)} \int_{S \backslash A} u_e\big(W_0^e + H - X + Y\big)\ dP \\
&s.t. \quad 0 \leqslant Y\mathbf{1}_{S \backslash A} \leqslant X\mathbf{1}_{S \backslash A} \\
&\int_{S \backslash A} (X - Y)\ dT \circ Q = 0
\end{aligned} \tag{B.2}$$

Remark B.1. *By the boundedness of u_e, the supremum of each of the above two problems is finite when their feasibility sets are nonempty. Now, the function X is feasible for problem (B.2), and so problem (B.2) has a nonempty feasibility set.*

Definition B.2. For a given $\beta \geqslant (1+\rho) H$, let $\Theta_{A,\beta}$ be the feasibility set of problem (B.1) with parameter β. That is,

$$\Theta_{A,\beta} := \left\{ Y \in B^{+}(\Sigma) : 0 \leqslant Y \leqslant X,\ \int (X - Y)\ dT \circ Q = \beta \right\}$$

Denote by Γ the collection of all β for which the feasibility set $\Theta_{A,\beta}$ is nonempty:

Definition B.3. Let $\Gamma := \left\{\beta \geqslant (1+\rho)\, H : \Theta_{A,\beta} \neq \varnothing\right\}$

Lemma B.4. *$\Gamma \neq \varnothing$.*

Proof. Choose $Y \in \mathcal{F}_{SB}$ arbitrarily, where $\mathcal{F}_{SB}$ is defined by equation (**??**). Then $Y \in B^+(\Sigma)$ is such that $0 \leqslant Y \leqslant X$, and $\int (X-Y)\ dT \circ Q \geqslant (1+\rho)\, H$. Let $\beta_Y = \int (X-Y)\ dT \circ Q$. Then, by definition of β_Y, and since $0 \leqslant Y \leqslant X$, we have $Y \in \Theta_{A,\beta_Y}$, and so $\Theta_{A,\beta_Y} \neq \varnothing$. Consequently, $\beta_Y \in \Gamma$, and so $\Gamma \neq \varnothing$. □

Lemma B.5. *X is optimal for problem (B.2).*

Proof. The feasibility of X for problem (B.2) is clear. To show optimality, let Y be any feasible solution for problem (B.2). Then for each $s \in S \backslash A$, $Y(s) \leqslant X(s)$. Therefore, since u_e is increasing, we have $u_e\big(W_0^e + H - X(s) + Y(s)\big) \leqslant u_e\big(W_0^e + H - X(s) + X(s)\big) = u_e\big(W_0^e + H\big)$, for each $s \in S \backslash A$. Thus,

$$\int_{S\backslash A} u_e\big(W_0^e + H - X + Y\big)\ dP \leqslant \int_{S\backslash A} u_e\big(W_0^e + H - X + X\big)\ dP = u\big(W_0^e + H\big) P\,(S\backslash A)$$

□

Remark B.6. *Since $Q\,(S\backslash A) = 0$ and $T\,(0) = 0$, it follows that $T \circ Q\,(S\backslash A) = 0$, and so $\int \mathbf{1}_{S\backslash A}\ dT \circ Q = T \circ Q\,(S\backslash A) = 0$, by Proposition A.6. Therefore, for any $Z \in B^+(\Sigma)$, it follows form the monotonicity and positive homogeneity of the Choquet integral (Proposition A.6) that*

$$0 \leqslant \int_{S\backslash A} Z\ dT \circ Q = \int Z \mathbf{1}_{S\backslash A}\ dT \circ Q \leqslant \int \|Z\|_s \mathbf{1}_{S\backslash A}\ dT \circ Q = \|Z\|_s \int \mathbf{1}_{S\backslash A}\ dT \circ Q = 0$$

and so $\int_{S\backslash A} Z\ dT \circ Q = 0$. Consequently, it follows form Proposition A.6 that for any $Z \in B^+(\Sigma)$,

$$\int Z\ dT \circ Q \leqslant \int Z \mathbf{1}_A\ dT \circ Q = \int_A Z\ dT \circ Q$$

Now, consider the following problem:

Problem B.7.

$$\sup_{\beta \in \Gamma} \left\{ F_A^*(\beta) : F_A^*(\beta) \ \textit{ is the supremum of problem (B.1), for a fixed } \beta \in \Gamma \right\}$$

Lemma B.8. *Under Assumption 3.1, if β^* is optimal for problem (B.7), and if Y_1^* is optimal for problem (B.1) with parameter β^*, then $Y^* := Y_1^* \mathbf{1}_A + X \mathbf{1}_{S\backslash A}$ is optimal for problem (**??**).*

Proof. By the feasibility of Y_1^* for problem (B.1) with parameter β^*, we have $0 \leqslant Y_1^* \leqslant X$ and $\int (X - Y_1^*)\ dT \circ P = \beta^*$. Therefore, $0 \leqslant Y^* \leqslant X$, and

$$\begin{aligned}\int (X - Y^*)\ dT \circ Q &= \int \left[(X - Y_1^*)\,\mathbf{1}_A + (X - X)\,\mathbf{1}_{S\backslash A}\right]\ dT \circ Q \\ &= \int_A (X - Y_1^*)\ dT \circ Q \geqslant \int (X - Y_1^*)\ dT \circ Q = \beta^* \geqslant (1+\rho)\,H\end{aligned}$$

where the inequality $\int_A (X - Y_1^*)\ dT \circ Q \geqslant \int (X - Y_1^*)\ dT \circ Q$ follows from the same argument as in Remark B.6. Hence, Y^* is feasible for problem (3.1). To show optimality of Y^* for problem (3.1), let $\overline{Y}$ be any other feasible function for problem (3.1), and define α by $\alpha = \int \left(X - \overline{Y}\right)\ dT \circ Q$. Then $\alpha \geqslant (1+\rho)\,H$, and so $\overline{Y}$ is feasible for problem (B.1) with parameter α, and α is feasible for problem (B.7). Hence

$$F_A^*(\alpha) \geqslant \int_A u_e\big(W_0^e + H - X + \overline{Y}\big)\ dP$$

Now, since β^* is optimal for problem (B.7), it follows that $F_A^*(\beta^*) \geqslant F_A^*(\alpha)$. Moreover, $\overline{Y}$ is feasible for problem (B.2) (since $0 \leqslant \overline{Y} \leqslant X$ and so $\int_{S\backslash A} \left(X - \overline{Y}\right)\ dT \circ Q = 0$ by Remark B.6). Thus,

$$\begin{aligned}F_A^*(\beta^*) + u_e\big(W_0^e + H\big)P(S\backslash A) &\geqslant F_A^*(\alpha) + u_e\big(W_0^e + H\big)P(S\backslash A) \\ &\geqslant \int_A u_e\big(W_0^e + H - X + \overline{Y}\big)\ dP + u_e\big(W_0^e + H\big)P(S\backslash A) \\ &\geqslant \int_A u_e\big(W_0^e + H - X + \overline{Y}\big)\ dP + \int_{S\backslash A} u_e\big(W_0^e + H - X + \overline{Y}\big)\ dP \\ &= \int u_e\big(W_0^e + H - X + \overline{Y}\big)\ dP\end{aligned}$$

However, $F_A^*(\beta^*) = \int_A u_e\big(W_0^e + H - X + Y_1^*\big)\ dP$. Therefore,

$$\int u_e\big(W_0^e + H - X + Y^*\big)\ dP = F_A^*(\beta^*) + u_e\big(W_0^e + H\big)P(S\backslash A) \geqslant \int u_e\big(W_0^e + H - X + \overline{Y}\big)\ dP$$

Hence, Y^* is optimal for problem (3.1). ∎

Remark B.9. *By Lemma B.8, we can restrict ourselves to solving problem* (B.1) *with a parameter* $\beta \in \Gamma$.

B.2. **Solving Problem (B.1).** Recall that for all $Z \in B(\Sigma)$, $\int Z\ dP = \int_A Zh\ dQ + \int_{S\backslash A} Z\ dP$, where $h = dP_{ac}/dQ$ is the Radon-Nikodým derivative of P_{ac} with respect to Q. Moreover, by definition of the set $A \in \Sigma$, we have $Q(S\backslash A) = P_s(A) = 0$. Therefore, $\int_A Zh\ dQ = \int Zh\ dQ$, for each $Z \in B(\Sigma)$. Hence, we can rewrite problem (B.1) (restricting ourselves to parameters $\beta \in \Gamma$ and recalling that $h = \phi \circ X$) as the following problem:

For a given $\beta \in \Gamma$,

$$\text{(B.3)} \qquad \begin{aligned} &\sup_{Y \in B(\Sigma)} \int u_e\big(W_0^e + H - X + Y\big)\ \phi(X)\ dQ \\ &s.t. \quad 0 \leqslant Y \leqslant X \\ &\qquad \int (X - Y)\ dT \circ Q = \beta \end{aligned}$$

Now, consider the following problem:

$$\text{(B.4)} \qquad \begin{aligned} &\text{For a given } \beta \in \Gamma, \\ &\sup_{Y \in B(\Sigma)} \int u_e\big(W_0^e + H - Z\big)\ \phi\left(X\right)\ dQ \\ &s.t. \quad 0 \leqslant Z \leqslant X \\ &\int Z\ dT \circ Q = \beta = \int_0^{+\infty} T\Big(Q\big(\{s \in S : Z\left(s\right) \geqslant t\}\big)\Big)\ dt \end{aligned}$$

Lemma B.10. *If Z^* is optimal for problem (B.4) with parameter β, then $Y^* := X - Z^*$ is optimal for problem (B.3) with parameter β.*

Proof. Let $\beta \in \Gamma$ be given, and suppose that Z^* is optimal for problem (B.4) with parameter β. Define $Y^* := X - Z^*$. Then $Y^* \in B\left(\Sigma\right)$. Moreover, since $0 \leqslant Z^* \leqslant X$, it follows that $0 \leqslant Y^* \leqslant X$. Now,

$$\int \left(X - Y^*\right)\ dT \circ Q = \int \Big(X - \left(X - Z^*\right)\Big)\ dT \circ Q = \int Z^*\ dT \circ Q = \beta$$

and so Y^* is feasible for problem (B.3) with parameter β. To show optimality of Y^* for problem (B.3) with parameter β, suppose, by way of contradiction, that $\overline{Y} \neq Y^*$ is feasible for problem (B.3) with parameter β and

$$\int u_e\big(W_0^e + H - X + \overline{Y}\big)\ h\ dQ > \int u_e\big(W_0^e + H - X + Y^*\big)\ h\ dQ$$

that is, with $\overline{Z} := X - \overline{Y}$, we have

$$\int u_e\big(W_0^e + H - \overline{Z}\big)\ h\ dQ > \int u_e\big(W_0^e + H - Z^*\big)\ h\ dQ$$

Now, since $0 \leqslant \overline{Y} \leqslant X$ and $\int \big(X - \overline{Y}\big)\ dT \circ Q = \beta$, we have that $\overline{Z}$ is feasible for problem (B.4) with parameter β, hence contradicting the optimality of Z^* for problem (B.4) with parameter β. Thus, $Y^* := X - Z^*$ is optimal for problem (B.3) with parameter β. □

Definition B.11. If $Z_1, Z_2 \in B^+\left(\Sigma\right)$ are feasible for problem (B.4) with parameter β, we will say that Z_2 is a Pareto improvement of Z_1 (or is Pareto-improving) when the following hold:

(1) $\int u_e\big(W_0^e + H - Z_2\big)\ h\ dQ \geqslant \int u_e\big(W_0^e + H - Z_1\big)\ h\ dQ$; and,
(2) $\int Z_2\ dT \circ Q \geqslant \int Z_1\ dT \circ Q$.

The next result shows that for any feasible claim for problem (B.4), there is a another feasible claim for problem (B.4), which is comonotonic with X and Pareto-improving.

Lemma B.12. *Fix a parameter $\beta \in \Gamma$. If Z is feasible for problem (B.4) with parameter β, then $\widetilde{Z}$ is feasible for probem (B.4) with parameter β, comonotonic with X, and Pareto-improving, where $\widetilde{Z}$ is the nondecreasing Q-rearrangement of Z with respect to X.*

Proof. Let Z be feasible for problem (B.4) with parameter β, and note that by Assumption 3.4, the map $\xi(X,Z) := u_e\big(W_0^e + H - Z\big)\ \phi(X)$ is supermodular (see Example A.10). Let $\widetilde{Z}$ denote the nondecreasing Q-rearrangement of Z with respect to X. Then by Lemma A.11 (2) and by equimeasurability of Z and $\widetilde{Z}$, the function $\widetilde{Z}$ is feasible for problem (B.4) with parameter β. Also, by Lemma A.11 (1) and by supermodularity of $\xi(X,Z)$, it follows that $\widetilde{Z}$ is Pareto-improving. $\square$

B.3. **Quantile reformulation.** Fix a parameter $\beta \in \Gamma$, let $Z \in B^+(\Sigma)$ be feasible for problem (B.4) with parameter β, and let $F_Z(t) = Q\big(\{s \in S : Z(s) \leqslant t\}\big)$ denote the distribution function of Z with respect to the probability measure Q, and let $F_X(t) = Q\big(\{s \in S : X(s) \leqslant t\}\big)$ denote the distribution function of X with respect to the probability measure Q. Let $F_Z^{-1}(t)$ be the left-continuous inverse of the distribution function F_Z (that is, the quantile function of Z), defined by

$$F_Z^{-1}(t) = \inf\Big\{z \in \mathbb{R}^+ : F_Z(z) \geqslant t\Big\},\ \forall t \in [0,1]$$

Let $\widetilde{Z}$ denote the nondecreasing Q-rearrangement of Z with respect to X. Since $Z \in B^+(\Sigma)$, it can be written as $\psi \circ X$ for some nonnegative Borel-measurable and bounded map ψ on $X(S)$. Moreover, since $0 \leqslant Z \leqslant X$, ψ is a mapping of $[0,M]$ into $[0,M]$. Let $\zeta := Q \circ X^{-1}$ be the image measure of Q under X. By Assumption 3.1, ζ is nonatomic. We can then define the mapping $\widetilde{\psi} : [0,M] \to [0,M]$ as in Appendix A.2 (see equation (A.1) on p. 11) to be the nondecreasing ζ-rearrangement of ψ, that is,

$$\widetilde{\psi}(t) := \inf\Big\{z \in \mathbb{R}^+ : \zeta\big(\{x \in [0,M] : \psi(x) \leqslant z\}\big) \geqslant \zeta\big(\,[0,t]\,\big)\Big\}$$

Then, as in Appendix A.2, $\widetilde{Z} = \widetilde{\psi} \circ X$. Therefore, for each $s_0 \in S$,

$$\widetilde{Z}(s_0) = \widetilde{\psi}(X(s_0)) = \inf\Big\{z \in \mathbb{R}^+ : \zeta\big(\{x \in [0,M] : \psi(x) \leqslant z\}\big) \geqslant \zeta\big(\,[0,X(s_0)]\,\big)\Big\}$$

However, for each $s_0 \in S$,

$$\zeta\big(\,[0,X(s_0)]\,\big) = Q \circ X^{-1}\big(\,[0,X(s_0)]\,\big) = F_X(X(s_0)) := F_X(X)(s_0)$$

Moreover,

$$\begin{aligned}\zeta\big(\{x \in [0,M] : \psi(x) \leqslant z\}\big) &= Q \circ X^{-1}\big(\{x \in [0,M] : \psi(x) \leqslant z\}\big)\\ &= Q\big(\{s \in S : \psi(X(s)) \leqslant z\}\big) = F_Z(z)\end{aligned}$$

Consequently, for each $s_0 \in S$,

$$\widetilde{Z}(s_0) = \inf\Big\{z \in \mathbb{R}^+ : F_Z(z) \geqslant F_X(X)(s_0)\Big\} = F_Z^{-1}(F_X(X(s_0))) := F_Z^{-1}(F_X(X))(s_0)$$

That is,

$$\widetilde{Z} = F_Z^{-1}(F_X(X)) \tag{B.5}$$

where F_Z^{-1} is the left-continuous inverse of F_Z, as defined in equation (3.2).

Hence, by Lemma B.12 and equation (B.5), we can restrict ourselves to finding a solution to problem (B.4) of the form $F^{-1}(F_X(X))$, where F is the distribution function of a function $Z \in B^+(\Sigma)$ such that $0 \leqslant Z \leqslant X$ and $\int Z\, dT \circ Q = \beta$. Moreover, since X is a nondecreasing function of X and Q-equimeasurable with X, it follows from the Q-a.s. uniqueness of the equimeasurable nondecreasing Q-rearrangement (see Appendix A.2) that $X = F_X^{-1}(F_X(X))$, Q-a.s. (see also [9, Lemma A.21]). Thus, for any $Z \in B^+(\Sigma)$,

$$\begin{aligned}\int u_e\big(W_0^e + H - F_Z^{-1}(F_X(X))\big)\ \phi\big(F_X^{-1}(F_X(X))\big)\ dQ &= \int u_e\big(W_0^e + H - \widetilde{Z}\big)\ \phi(X)\ dQ \\ &\geqslant \int u_e\big(W_0^e + H - Z\big)\ \phi(X)\ dQ\end{aligned}$$

where the inequality follows from the proof of Lemma B.12. Moreover, since $\zeta = Q \circ X^{-1}$ is nonatomic (by Assumption 3.1), it follows that $F_X(X)$ has a uniform distribution over $(0,1)$ [9, Lemma A.21], that is, $Q\big(\{s \in S : F_X(X)(s) \leqslant t\}\big) = t$ for each $t \in (0,1)$. Finally, letting $U := F_X(X)$,

$$\begin{aligned}\int F^{-1}(U)\ dT \circ Q &= \int_0^{+\infty} T\Big[Q\big(\{s \in S : F^{-1}(U)(s) \geqslant t\}\big)\Big]\ dt \\ &= \int_0^{+\infty} T\Big[Q\big(\{s \in S : F^{-1}(U)(s) > t\}\big)\Big]\ dt \\ &= \int_0^{+\infty} T\Big[1 - F(t)\Big]\ dt \\ &= \int_0^1 T'(1-t)\,F^{-1}(t)\ dt = \int T'(1-U)\,F^{-1}(U)\ dQ\end{aligned}$$

where the third and last equalities above follow from the fact that U has a uniform distribution over $(0,1)$, and where the second-to-last equality follows from a standard argument[5].

Now, recall from Definition 3.3 that $\mathcal{A}Quant$ given in equation (3.3) is the collection of all *admissible quantile functions*, that is the collection of all functions f of the form F^{-1}, where F is the distribution function of a function $Z \in B^+(\Sigma)$ such that $0 \leqslant Z \leqslant X$, and consider the following problem:

$$\text{(B.6)}\qquad \begin{aligned}&\text{For a given } \beta \in \Gamma \\ &\qquad \sup_f\ V(f) = \int u_e\big(W_0^e + H - f(U)\big)\ \phi\left(F_X^{-1}(U)\right)\ dQ \\ &\qquad s.t. \quad f \in \mathcal{A}Quant \\ &\qquad\qquad \int T'(1-U)\,f(U)\ dQ = \beta\end{aligned}$$

Lemma B.13. *If f^* is optimal for problem (B.6) with parameter $\beta \in \Gamma$, then the function $f^*(U)$ is optimal for problem (B.4) with parameter β, where $U := F_X(X)$. Moreover, $X - f^*(U)$ is optimal for problem (B.3) with parameter β.*

Proof. Fix $\beta \in \Gamma$, suppose that $f^* \in \mathcal{A}Quant$ is optimal for problem (B.6) with parameter β, and let $Z^* \in B^+(\Sigma)$ be the corresponding function. That is, f^* is the quantile function of Z^* and $0 \leqslant Z^* \leqslant X$. Let $\widetilde{Z}^* := f^*(U)$. Then $\widetilde{Z}^*$ is the equimeasurable nondecreasing Q-rearrangement of

[5] See, e.g. Denneberg [7], Proposition 1.4 on p. 8 and the discussion on pp. 61-62. See also [19, p. 418], [16, p. 210, p. 213], or [3, p. 207].

Z^* with respect to X, and so $0 \leqslant \widetilde{Z}^* \leqslant X$ by Lemma A.11 (2). Moreover,

$$\begin{aligned}\beta &= \int T'(1-U) f^*(U)\ dQ = \int f^*(U)\ dT \circ Q \\ &= \int \widetilde{Z}^*\ dT \circ Q = \int_0^{+\infty} T\Big[Q\big(\{s \in S : \widetilde{Z}^*(s) \geqslant t\}\big)\Big]\ dt \\ &= \int_0^{+\infty} T\Big[Q\big(\{s \in S : Z^*(s) \geqslant t\}\big)\Big]\ dt = \int \widetilde{Z}^*\ dT \circ Q\end{aligned}$$

where the second-to-last equality follows from the Q-equimeasurability of Z^* and $\widetilde{Z}^*$. Therefore, $\widetilde{Z}^* = f^*(U)$ is feasible for problem (B.4) with parameter β. To show optimality, let Z be any feasible solution for problem (B.4) with parameter β, and let F be the distribution function for Z. Then, by Lemma B.12, the function $\widetilde{Z} := F^{-1}(U)$ is feasible for probem (B.4) with parameter β, comonotonic with X, and Pareto-improving. Moreover, $\widetilde{Z}$ has also F as a distribution function. To show optimality of $\widetilde{Z}^* = f^*(U)$ for problem (B.4) with parameter β it remains to show that

$$\int u_e\big(W_0^e + H - \widetilde{Z}^*\big)\ \phi(X)\ dQ \geqslant \int u_e\big(W_0^e + H - \widetilde{Z}\big)\ \phi(X)\ dQ$$

Now, let $f := F^{-1}$, so that $\widetilde{Z} = f(U)$. Since $\widetilde{Z}$ is feasible for probem (B.4) with parameter β, we have

$$\begin{aligned}\beta &= \int \widetilde{Z}\ dT \circ Q = \int F^{-1}(U)\ dT \circ Q \\ &= \int_0^1 T'(1-t) F^{-1}(t)\ dt = \int T'(1-U) f(U)\ dQ\end{aligned}$$

Hence, f is feasible for problem (B.6) with parameter β. Since f^* is optimal for problem (B.6) with parameter β we have

$$\int u_e\big(W_0^e + H - f^*(U)\big)\ \phi\big(F_X^{-1}(U)\big)\ dQ \geqslant \int u_e\big(W_0^e + H - f(U)\big)\ \phi\big(F_X^{-1}(U)\big)\ dQ$$

Finally, since $X = F_X^{-1}(U)$, Q-a.s., we have

$$\int u_e\big(W_0^e + H - \widetilde{Z}^*\big)\ \phi(X)\ dQ \geqslant \int u_e\big(W_0^e + H - \widetilde{Z}\big)\ \phi(X)\ dQ$$

Therefore, $\widetilde{Z}^* = f^*(U)$ is optimal for problem (B.4) with parameter β. Finally, by Lemma B.10, $Y^* := X - \widetilde{Z}^* = X - f^*(U)$ is optimal for problem (B.3) with parameter β. □

By Lemmata B.8 and B.13, this completes the proof of Theorem 3.5.

Appendix C. Proof of Corollary 3.8

Recall from equation (3.3) that

$$\mathcal{A}Quant = \Big\{f \in Quant : 0 \leqslant f(z) \leqslant F_X^{-1}(z),\ \text{for each } 0 < z < 1\Big\},$$

where $Quant = \left\{ f : (0,1) \to \mathbb{R} \;\middle|\; f \text{ is nondecreasing and left-continuous} \right\}$. Define the collection $\mathcal{K}$ of functions on $(0,1)$ as follows:

$$\mathcal{K} = \left\{ f : (0,1) \to \mathbb{R} \;\middle|\; 0 \leqslant f(z) \leqslant F_X^{-1}(z), \text{ for each } 0 < z < 1 \right\} \tag{C.1}$$

Then $\mathcal{A}Quant = Quant \cap \mathcal{K}$. Consider the following problem, with parameter $\beta \in \Gamma$:

$$\begin{aligned} &\text{For a given } \beta \in \Gamma \\ &\qquad \sup_f \; V(f) = \int_0^1 u_e\big(W_0^e + H - f(t)\big)\; \phi\left(F_X^{-1}(t)\right)\; dt \\ &\qquad s.t. \quad f \in \mathcal{A}Quant \\ &\qquad\qquad \int_0^1 T'(1-t)\, f(t)\; dt = \beta \end{aligned} \tag{C.2}$$

Lemma C.1. *For a given $\beta \in \Gamma$, if $f^* \in \mathcal{A}Quant$ satisfies the following:*

(1) $\int_0^1 T'(1-t)\, f^*(t)\; dt = \beta$;

(2) *There exists $\lambda \leqslant 0$ such that for all $t \in (0,1) \setminus \{t : \phi \circ F_x^{-1}(t) = 0\}$,*

$$f^*(t) = \underset{0 \leqslant y \leqslant F_X^{-1}(t)}{\arg\max} \left[u_e\left(W_0^e + H - y\right) \phi\left(F_X^{-1}(t)\right) - \lambda T'(1-t)\, y \right]$$

Then f^* solves problem (C.2) with parameter β

Proof. Fix $\beta \in \Gamma$, suppose that $f^* \in \mathcal{A}Quant$ satisfies conditions (1) and (2) above. Then, in particular, f^* is feasible for problem (C.2) with parameter β. To show optimality of f^* for problem (C.2) with parameter β, let f by any other feasible solution for problem (C.2) with parameter β. Then, for all $t \in (0,1) \setminus \{t : \phi \circ F_x^{-1}(t) = 0\}$,

$$\begin{aligned} &u_e\left(W_0^e + H - f^*(t)\right) \phi\left(F_X^{-1}(t)\right) - \lambda T'(1-t)\, f^*(t) \\ &\qquad\qquad \geqslant u_e\left(W_0^e + H - f(t)\right) \phi\left(F_X^{-1}(t)\right) - \lambda T'(1-t)\, f(t) \end{aligned}$$

That is, $\Big[u_e\left(W_0^e + H - f^*(t)\right) - u_e\left(W_0^e + H - f(t)\right)\Big] \phi\left(F_X^{-1}(t)\right) \geqslant \lambda T'(1-t)\Big[f^*(t) - f(t)\Big]$. Integrating yields $V(f^*) - V(f) \geqslant \lambda\left[\beta - \beta\right] = 0$, that is $V(f^*) \geqslant V(f)$, as required. □

Hence, in view of Lemma C.1, in order to find a solution for problem (C.2) with a given parameter $\beta \in \Gamma$ and a given $\lambda \leqslant 0$, one can start by solving the problem

$$\max_{0 \leqslant f_\lambda(t) \leqslant F_X^{-1}(t)} \left[u_e\left(W_0^e + H - f_\lambda(t)\right) \phi\left(F_X^{-1}(t)\right) - \lambda T'(1-t)\, f_\lambda(t) \right] \tag{C.3}$$

for a fixed $t \in (0,1) \setminus \{t : \phi \circ F_x^{-1}(t) = 0\}$.

Consider first the following problem:

$$\max_{f_\lambda(t)} \left[u_e\left(W_0^e + H - f_\lambda(t)\right) \phi\left(F_X^{-1}(t)\right) - \lambda T'(1-t)\, f_\lambda(t) \right] \tag{C.4}$$

for a fixed $t \in (0,1) \setminus \{t : \phi \circ F_x^{-1}(t) = 0\}$.

By concavity of the utility function u, in order to solve problem (C.4), it suffices to solve for the first-order condition

$$-u_e'\left(W_0^e + H - f_\lambda^*(t)\right)\phi\left(F_X^{-1}(t)\right) - \lambda T'(1-t) = 0$$

which gives

$$f_\lambda^*(t) = W_0^e + H - \left(u_e'\right)^{-1}\left(\frac{-\lambda T'(1-t)}{\phi\left(F_X^{-1}(t)\right)}\right) \tag{C.5}$$

Then the function $f_\lambda^*(t)$ solve problem (C.4), for a fixed $t \in (0,1)\backslash\{t : \phi \circ F_x^{-1}(t) = 0\}$.

By Assumption 3.7, the function $t \mapsto \frac{T'(1-t)}{\phi\left(F_X^{-1}(t)\right)}$ is nondecreasing. By Assumption 2.1 , the function u_e is strictly concave and continuously differentiable. Hence, the function u_e' is both continuous and strictly decreasing. This then implies that $\left(u_e'\right)^{-1}$ is continuous and strictly decreasing, by the Inverse Function Theorem [25, pp. 221-223]. Therefore, the function $f_\lambda^*(t)$ in equation (C.5) is nondecreasing ($\lambda \leqslant 0$). Moreover, by Assumption 3.1 and Assumption 3.6, $f_\lambda^*(t)$ is left-continuous.

Define the function f_λ^{**} by

$$f_\lambda^{**}(t) = \max\left[0, \min\left\{F_X^{-1}(t), f_\lambda^*(t)\right\}\right] \tag{C.6}$$

Then $f_\lambda^{**}(t) \in \mathcal{K}$. Moreover, since both F_X^{-1} and f_λ^* are nondecreasing and left-continuous functions, it follows that f_λ^{**} is nondecreasing and left-continuous. Consequently, $f_\lambda^{**}(t) \in \mathcal{A}Quant$. Finally, it is easily seen that $f_\lambda^{**}(t)$ solves problem (C.3) for the given λ. Now, for a given $\beta_0 \in \Gamma$, if λ^* is chosen so that $\int_0^1 T'(1-t) f_{\lambda^*}^{**}(t)\ dt = \beta_0$, then by Lemma C.1, $f_{\lambda^*}^{**}$ is optimal for problem (C.2) with parameter β_0.

Hence, to conclude the proof of Corollary 3.8, it remains to show that for each $\beta_0 \in \Gamma$, there exists a $\lambda^* \leqslant 0$ such that $\int_0^1 T'(1-t) f_{\lambda^*}^{**}(t)\ dt = \beta_0$. This is given by Lemma C.2 below.

Lemma C.2. *Let ψ be the function of the parameter $\lambda \leqslant 0$ defined by $\psi(\lambda) := \int_0^1 T'(1-t) f_\lambda^{**}(t)\ dt$. Then for each $\beta_0 \in \Gamma$, there exists a $\lambda^* \leqslant 0$ such that $\psi(\lambda^*) = \beta_0$.*

Proof. First note that ψ is a continuous and nonincreasing function of λ, where continuity of ψ is a consequence of Lebesgue's Dominated Convergence Theorem [1, Theorem 11.21]. Indeed, since X is bounded and since F_X^{-1} is nondecreasing, it follows that for each $t \in [0,1]$,

$$\min\left\{F_X^{-1}(t), f_\lambda^*(t)\right\} \leqslant F_X^{-1}(t) \leqslant F_X^{-1}(1) \leqslant M = \|X\|_s < +\infty.$$

Moreover, since T is concave and increasing, T' is nonincreasing and nonnegative, and so for each $t \in [0,1]$, $0 \leqslant T'(1-t) \leqslant T'(0)$. But $T'(0) < +\infty$, by Assumption 3.1. Hence, for each $t \in [0,1]$,

$$\min\left\{F_X^{-1}(t), f_\lambda^*(t)\right\} T'(1-t) \leqslant F_X^{-1}(1)\, T'(0) \leqslant \|X\|_s\, T'(0) < +\infty$$

Moreover, $\psi(0) = 0$ (by Assumption 2.1), and

$$\begin{aligned}\lim_{\lambda \to -\infty} \psi(\lambda) &= \int_0^1 T'(1-t) \min\left\{F_X^{-1}(t), W_0^e + H\right\} dt \\ &= \int_0^{F_X\left(W_0^e+H\right)} T'(1-t) F_X^{-1}(t)\ dt + \left(W_0^e + H\right) \int_{F_X\left(W_0^e+H\right)}^1 T'(1-t)\ dt\end{aligned}$$

However, by Assumption 3.2, we have $F_X\left(W_0^e + H\right) = 1$. This then implies that

$$\lim_{\lambda \to -\infty} \psi(\lambda) = \int_0^1 T'(1-t) F_X^{-1}(t)\ dt = \int X\ dT \circ Q$$

Now, for any $\beta_0 \in \Gamma$, and for any $Y \in B^+(\Sigma)$ which is feasible for problem (B.1) with parameter β_0, one has:

(i) $0 \leqslant Y \leqslant X$; and,

(ii) $\int (X - Y)\ dT \circ Q = \beta_0$.

Hence, $0 \leqslant X - Y \leqslant X$, and so, by monotonicity of the Choquet integral (Proposition A.6), it follows that $\beta_0 = \int (X - Y)\ dT \circ Q \leqslant \int X\ dT \circ Q$. Consequently, for any $\beta_0 \in \Gamma$,

$$0 = \psi(0) \leqslant \beta_0 \leqslant \int X\ dT \circ Q = \lim_{\lambda \to -\infty} \psi(\lambda)$$

Hence, by the Intermediate Value Theorem [25, Theorem 4.23], for each $\beta_0 \in \Gamma$ there exists some $\lambda^* \leqslant 0$ such that $\psi(\lambda^*) = \beta_0$. $\square$

By Lemmata C.1 and C.2, this concludes the proof of Corollary 3.8.

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