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Contracting for Innovation under Ambiguity

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At any given point in time, the collection of assets that exist in the economy is observable. Each asset is a function of a set of contingencies. The union taken over all assets of these contingencies is what we call the *set of publicly known states*. An innovation is a set of states that are not publicly known along with an asset (in a broad sense) that pays contingent on those states. The creator of an innovation is an entrepreneur. He is represented by a probability measure on the set of new states. All other agents perceive the innovation as *ambiguous*: each of them is represented by a set of probabilities on the new states. The agents in the economy are classified with respect to their attitude toward the Ambiguity: the financiers are (locally) ambiguity seeking while the consumers are ambiguity averse. An entrepreneur and a financier come together when the former seeks funds to implement his project and the latter seeks new profit opportunities. The resulting contracting problem does not fall within the standard theory due to the presence of Ambiguity (on the financier's side) and to the heterogeneity in the parties' beliefs. We prove existence and monotonicity (i.e., truthful revelation) of the optimal contract. We characterize this contract under the additional assumption that the financiers are globally ambiguity seeking. Finally, we reformulate our results in an insurance framework and extend the classical result of Arrow-Borch-Raviv and the more recent one of Ghossoub. In the case of an Ambiguity averse insurer, we also show that the optimal contract has the form of a generalized deductible.

Keywords: Innovation, entrepreneur, financier, ambiguity, contracts, monotonicity, deductible.

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I. INTRODUCTION

In this paper, we study the problem of contracting for innovation between an entrepreneur and a financier. In the first a third of the paper, we address the following questions: What does "innovation" mean? Are "entrepreneur" and "financier" just two labels or is there something substantial behind these denominations? Why does "contracting for innovation" differ from other contracting problems?

Two strands of literature merge in our work: the literature on entrepreneurship and innovation and the literature on contracting under ambiguity. We contribute to the former by building a theoretical framework where we can answer the questions raised above; we contribute to the latter by studying and solving a novel problem of contracting under ambiguity.

The paper is organized as follows. We present our ideas on entrepreneurship and innovation in Sections I A to IV. Section II, in particular, contains the formal definition of Innovation that we introduce in this paper. Section IV concludes this part by briefly discussing our contribution in relation to some of the existing literature. The ideas elaborated in these sections lead to the formulation of a certain contracting problem in Section V. We discuss some related literature on contracts in Section VI. In Section VII, we state our theorem on the existence and monotonicity of an optimal contract. We characterize this contract under an additional assumption in Section VIII. In the final section, we observe that – with some technical changes – our result can be re-interpreted in an insurance framework, and compare it to the classical one of Arrow-Borch-Raviv ([4], [7] and [31]). Two Appendices, containing some background material and the proofs omitted from the main text, complete the exposition.

A. The inadequacy of the classical model

Our interest in the problem of contracting for innovation is rooted in a broad project ([29], [30], [2], [22], [5] [3]) which aims at answering the following questions: How do capitalist systems generate their dynamism? and, Why is a capitalist economy inherently different from a centrally planned one? Our research has been inspired by the fundamental belief that in order to study these issues, we must study the mechanisms of entrepreneurship and innovation in capitalist economies: the role of entrepreneurs in seeing commercial possibilities for developing and adopting products that exploit new technologies; the role of entrepreneurs in conceiving and developing new products and methods; the role of financiers in identifying entrepreneurs to back and to advise; and the incentives and disincentives for entrepreneurship inside established corporations. This means studying both the entrepreneur as a micro actor and the entrepreneurial economy as an interactive system.

Thus, we do believe that entrepreneurs and financiers are special types of economic agents and that the process of innovation plays a fundamental role in explaining the dynamics of capitalist economies. Yet, this belief clashes against some of the fundamental constructions of economic theory. Think of the Arrow-Debreu model: any equilibrium outcome achievable in a decentralized economy can also be achieved in a centrally planned one, anybody can be a financier, and there is no profit to be made with this activity anyway. By introducing frictions in the Arrow-Debreu model, such as frictions in the financial markets for example, we could make sense of the notion of *financier* by appealing to differences in the agents' initial endowments. Yet, this would not explain why certain financiers are successful while others are not: after all, in an equilibrium of the model, they all share the same view and have the same opportunities. And, what is an entrepreneur in this model?

We believe that the main drawbacks of the classical theories do not reside in an excessive idealization of actual economies. We believe, instead, that those drawbacks reside in a fundamental modeling issue: the way Uncertainty is treated. We contend that it is precisely the classical treatment of Uncertainty that has led to a gross misrepresentation of the role played by entrepreneurs and financiers in actual economies. It is to this treatment and to our proposed remedy that we move next.

B. Objective states vs Subjective states

When dealing with uncertainty, a central concept in classical theories is that of *state of the world*. Following the Bayesian tradition, a state of the world is a complete specification of all the parameters defining an environment. For instance, a state of the world for the economy would consist of a specification of temperature, humidity, consumers' tastes, technological possibilities, detailed maps of all possible planets, etc. According to this view, the future is uncertain because it is not known in advance which state will obtain. In principle (but this is clearly beyond human capabilities), one might come up with the full list of all possible states, and classical theories postulate that each and every agent would be described by a probability measure on such a list. While (ex ante) different agents might have different views (i.e., different probability distributions), the information conveyed by the market eventually leads them to entertain the same view: at an economy's equilibrium no two agents are willing to bet against each other about the uncertainty's resolution. Thus, in classical theories, there is nothing uncontroversial about the way one deals with uncertainty.

The contrast between this prediction of the theory and what happens in real life is striking. Actual economic agents usually disagree about the resolution of uncertainty, and even the assumption of a list of contingencies known to all agents as well as the assumption of each agent having a probability distribution over such contingencies seem hardly tenable. It is an old idea, dating back to at least F. Knight, that some – and, perhaps, the most relevant – economic decisions are made under circumstances where the information available is too coarse to make full sense of the surrounding environment, where things look too fuzzy to have a probability distribution over a set of relevant contingencies. In such situations, Risk Theory is simply of no use. We fully adhere to this view.

The concept of state of the world is central to our theory as well. We depart, however, from classical theories in that we do not assume the existence of a list of all possible states which is known to all agents. We do so for several reasons. First, we believe that this assumption is too artificial. Second, a theory built on such an assumption would not be testable, not even in principle. Third, and more importantly, we believe that, by making such an assumption, we would lose sight of the actual role played by entrepreneurs and financiers in actual economies.

We take a point of view that we might deem "objective". We take off from the (abstract) notion of asset. In its broadest interpretation, an asset is, by definition, something that pays off depending on the realization of certain contingencies. In other words, in order to define an asset, one must specify a list of contingencies along with the amount that the asset pays as a function of those. At each point in time, the set of assets that exist in the economy is observable. Thus, in principle, the set of contingencies associated with each asset is objectively given. The union, taken over all the assets, of all these contingencies is then objectively given, in the sense that is it derived from observables. We call this set *the set of publicly known states of the world*, and denote it by SP. We assume that each and every agent in the economy is aware of all the states contained in SP. We stress, however, that what is more important is that this set be knowable rather than be known by every agent.

Of course, there is no reason why each agent in the economy, individually considered, be restricted to hold the same view. In other words, while we assume that each agent is aware of the set SP, we are also open to the possibility that each agent might consider states that are not in SP. Formally, we admit that each agent i has a subjective state space S_i of the form $S_i = SP \cup I_i$, where I_i is the list of contingencies in agent i's set of states that are not publicly known. An example might clarify. In the '50s, IBM was investing in the creation of (big) computers. In our terminology, this means that IBM had envisioned states of the world where computers would be produced and sold, where hardware and software for computers would be produced and sold, etc. Since IBM stocks were tradable, these states would be part of the publicly known states according to our definition. Some time between the late '50s and the early '60s, Doug Englebart envisioned a world where a PCs existed and where software and hardware for PCs would be produced and sold. According to our view, before Doug Englebart began patenting his ideas, these states existed only in his mind (and maybe in those of few others), that is they were part of Doug Englebart's subjective state space, but they were not publicly knowable.

We are going to assume that each agent i is a Bayesian decision maker with respect to his own subjective state space. That is, agent i with subjective state space S_i makes his decisions according to a probability distribution P_i on S_i. In the terminology that we will be using in Section III, this means that we assume that each agent believes that he has a good understanding of his own state spaces. While this assumption could be removed, we believe that it is a good first approximation. Moreover, we believe that it follows quite naturally from the idea of subjective state space, as we define it.

II. INNOVATION

The idea of "innovation" and the way we model it is central to our theory. Unquestionably, the ability to "innovate" is one of the most distinguishing features of capitalist economies. Innovations occur in the form of new consumption goods, new technological processes, new institutions, new forms of organizations in trading activities, etc. We abstract from the differences existing across different types of innovation, and focus on what is common among them. For us, an innovation is defined as follows:

Definition 1 *An innovation is a set of states of the world which are not publicly known along with an asset which pays contingent on those states.*

An example will clarify momentarily. For now, we would like to point out that the word "asset" in the definition should be interpreted in a broad sense. That is, by asset we mean any activity capable of generating economic value. An innovation will be denoted by a pair (S_j, X_j), where j is the innovator, S_j is his subjective state space and X_j is the asset that pays contingent on states in S_j. Notice that, as it is encoded in the definition of the subjective state space S_j, we allow for X_j to also pay off contingent on states in SP.

In order to illustrate the definition, let us imagine an economy where historically only two types of cakes have been consumed: carrot cakes and coconut cakes. Each year, each individual consumer might be of one of two types: either he likes carrot cakes (consumers of type 1) or coconut cakes (consumers of type 2) but not both. The fraction of the population made of consumers of type 1 varies from year to year according to some known stochastic process. Summing up, in our economy there are two productive processes: one for producing carrot cakes and one for coconut cakes. There is a continuum of tomorrow's states, where each state gives the fraction of consumers of type 1. These states are understood by everyone in the economy. That is, $SP = [0, 1]$ and a point x in $[0, 1]$ means that the fraction of type 1 consumers is x. Moreover, there is a given probability distribution on $[0, 1]$, which is known to everyone in the economy.

Now, suppose that an especially creative individual, whom we call e, comes into the scene and (a) figures out a new productive process that produces banana cakes; (b) believes that each consumer, whether of type 1 or 2, would switch to banana cakes with probability $1/3$ if given the opportunity. What is happening here is that agent e has: (1) imagined a whole set of new states, those in which consumers might like banana cakes (in fact, the subjective state space for agent e is two-dimensional, while SP is one-dimensional); (2) imagined that a non-negligible probability mass might be allocated to the extra dimension conditional on the consumers being given the chance to consume banana cakes; and, (3) figured out a device (the productive process) that makes the new states capable of generating economic value.

Hopefully, the example has convincingly demonstrated that the definition given above is the "right" definition in that it conveys the essential features which identify any innovation (the new states along with the new activity). We believe that one of its virtues is that it makes it clear that the process of innovation is truly associated to the appearance of new and fundamentally different possibilities: from the viewpoint of the innovator, both the state space and the space of production possibilities have higher dimensionality.

Definition 2 *An agent e who issues an innovation is called an entrepreneur.*

Recall that we assumed that each agent has a probability distribution on his subjective state space. Thus, an entrepreneur is described by a triple (S_e, X_e, P_e), where (S_e, X_e) is the innovation and P_e is his subjective probability on the subjective state space S_e.

III. UNCERTAINTY AND THE CLASSIFICATION OF ECONOMICS AGENTS

In our story, the innovators are the entrepreneurs. But what is going to happen once they come up with an innovation? In the economy above, how are the consumers going to react if they are told that banana cakes will be available? We follow up on the idea that an innovation is associated to a new scenario, something that the economy as a whole has never experienced. It is then natural to regard such a situation as one of Knightian uncertainty (or Ambiguity): the information available is (except, possibly, for the entrepreneur) too coarse to form a probability distribution on the relevant contingencies. Notice that Ambiguity enters our model in a rather novel way: its source is not some device (Nature) outside the economic system; rather, it is some of the economic actors – the entrepreneurs – who are the primary source of Ambiguity.

Decision theorists have developed several models to deal with this problem, all of which stipulate that the behavior of agents facing Ambiguity is described not by a single probability but rather by a set of probabilities (see [19] for a comprehensive survey). Formally, the problem is as follows. Let e be an entrepreneur, and let i be another agent. Agent i is represented by a pair (S_i, P_i), where S_i is his subjective state space and P_i a probability on S_i. Suppose that i has never thought of the subjective states of the entrepreneur. Now, suppose that agent i is made aware, directly or otherwise, of the innovation (S_e, X_e) as well as of the probability P_e of the entrepreneur. What is i going to do? Is he going to believe e and adopt his view (i.e., the probability P_e) or is i going to form a different opinion? Is i going to form an opinion at all? Clearly, each of these cases is possible and there is no real reason to favor one over the other. Thus, we need a way to model all these possibilities simultaneously. We are going to do so as follows. When agent i becomes aware of the subjective states of agent e, the set of states for agent i becomes $S_i \cup S_e$. Thus, agent i's problem is that of extending his view from S_i to the union $S_i \cup S_e$ as this is necessary for evaluating assets that pay contingent on S_e. We assume that agent i makes this extension by using all the probability distributions on $S_i \cup S_e$ which are compatible with his original view, that is all those probabilities on $S_i \cup S_e$ whose conditional on S_i is P_i. The exact way in which agent i will evaluate the assets defined on S_e depends, loosely speaking, on the way all these probabilities are aggregated and, in general, different agents will aggregate them in different ways. Put in a different terminology, an agent's evaluation of the assets defined on S_e depends on the agent's attitude toward Ambiguity. This observation suggests a natural classification of economic agents: in one category we would put those agents who are going to share, at least partially, the view of at least one entrepreneur while in the other we would put those who are not going to do so under any circumstances. The former have the potential to become business partners of some entrepreneurs, the latter will never do so. Thus, we are going to distinguish between *consumers* and *financiers* that are defined as follows.

Consumers Their subjective state space coincides with the publicly known set of states. They are ambiguity averse, in the sense that they always evaluate their options according to the worst probability (worst case scenario = maxmin expected utility). Formally, a consumer c is represented by a pair (SP, P_c); when facing an innovation (S_e, X_e), c evaluates it by using the functional

$$C(X_e) = \min_{P \in \mathcal{C}_c} \int u_c(X_e) dP$$

where $\mathcal{C}_c$ is the set of all probabilities on $SP \cup S_e$ whose conditional on SP is P_c and u_c is the consumer's utility on outcomes.

Notice that this description easily implies that (a) if there exists a bond in the economy, and (b) if there exists a state in S_e such that the worth of the innovation is below the bond, then the consumer will not buy that innovation at any positive price. Under these circumstances, these agents will never become business partners of any entrepreneur, which explain why we call them consumers.

Financiers Their subjective state space coincides with the publicly known set of states. They are less ambiguity averse than the consumers. A financier φ is represented by a pair (SP, P_φ); when facing an innovation (S_e, X_e), φ evaluates it by using the functional

$$\Phi(X_e) = \alpha(X_e) \min_{Q \in \mathcal{C}_\varphi} \int u_\varphi(X_e) dQ + (1 - \alpha(X_e)) \max_{Q \in \mathcal{C}_\varphi} \int u_\varphi(X_e) dQ \tag{1}$$

where $\mathcal{C}_\varphi$ is the set of probabilities on $SP \cup S_e$ whose conditional on SP is P_φ and u_φ is the financier's utility on outcomes. For each asset X_e, the coefficient $\alpha(X_e) \in [0, 1]$.

Thus, the functional (1) is a combination of aversion toward projects that involve new states (the *min* part of the functional) and lean toward the same projects (the *max* part). Intuitively, the coefficient $\alpha(X_e)$ represents the degree of Ambiguity aversion of the financier (see [15, 16]), and this degree is allowed to vary with the asset (= entrepreneurial project) to be evaluated. We suppose that for at least one asset X_e, $\alpha(X_e) < 1$. A special case obtains when the financiers' attitude toward Ambiguity does not depend on the project to be evaluated. In such a case, projects are evaluated by using the functional

$$\Phi(X_e) = \alpha \min_{Q \in \mathcal{C}_\varphi} \int u_\varphi(X_e) dQ + (1 - \alpha) \max_{Q \in \mathcal{C}_\varphi} \int u_\varphi(X_e) dQ$$

where we suppose that $\alpha < 1$.

We believe that our categorization captures the essential (functional) distinction between the concept of "consumer" and "financier": a (pure) consumer is someone who rejects the unknown, while a financier is somebody that is willing to bet on it. The condition in the above definitions that both the consumer's and the financier's state space is SP only means that consumers and financiers are not entrepreneurs. One might argue that this assumption is natural in the case of consumers but it is not so in the case of financiers. This is not problematic as a financier's subjective state space bigger than SP can be easily accommodated in our framework by suitably re-defining the function $\alpha(X_e)$, which represents the financier's Ambiguity aversion.

In sum, we have three types of agent: entrepreneurs, financiers and consumers. The study of economies populated by these types of agents (the way we defined them) poses entirely new problems. Here, since we are concerned with the problem of contracting between financiers and entrepreneurs, we leave it at that. We refer the interested reader to [3] for a preliminary inquiry into the properties of these economies.

IV. COMMENTS AND RELATED LITERATURE

The literature on innovation is vast. Spanning from Schumpeter [37] to the works of Reinganum [32], Roemer [33], Scotchmer [38] and Boldrin and Levine [6], it contains many more important papers than we could reasonably cite. We refer to [34] for a comprehensive list of references. It is probably fair to say that most of these works have focused on a particular aspect of innovation or on a particular role played by it, a choice usually dictated by the problem under study. Our definition is an attempt to account simultaneously for all those aspects. We hope that, in such a way, it will appear as a concept that can easily be exported and particularized to any setting where the intuitive idea of innovation might play a significant role.

Undoubtedly, our construction has a strong Schumpeterian flavor: for instance, the entrepreneur is the creator of the innovation[1], the entrepreneur is a singular actor, our financiers are quite like Schumpeter's bankers, the functional classification of the economic agents, etc. Clearly, there are considerable differences as well. The most notable is in the definition of innovation: ours is a far

[1] Schumpeter distinguishes between those who create ideas and those, the entrepreneurs, who turn them into something of economic value. Roughly, in our model this would correspond to distinguishing between those who come up with the new states (inventors) and those who make those states suitable of generating economic value (entrepreneurs) by issuing assets that pay contingent on those states.

reaching generalization of Schumpeter's notion, which consists only of a new combination of the inputs in the productive process. Another difference worth stressing is the following. Schumpeter's work, as it is well-known, is regarded as a celebration of the entrepreneur: this is viewed as a privileged individual that in a condition of severe uncertainty (the newly thought states) has a "vision" (the project/asset) that might change the course of the economy[2]. While this is true in our construction as well, the appearance of this "vision" would be rather inconsequential if it were not coupled with another "vision", that of the financier. In our construction, the vision of the entrepreneur leads to the appearance of Ambiguity. It is only the insight of the financier in this Ambiguity that recognizes the vision of the entrepreneur and makes the change possible. Formally, this insight appears in the form of the coefficient $\alpha(X_e)$ being low enough, which means precisely that the financier believes in the profitability of the entrepreneur's project.

V. CONTRACTING FOR INNOVATION

All that we have said so far leads to the following problem. An entrepreneur comes up with a new idea. Not having enough wealth to implement it, he goes to a financier and describes his project, hoping to obtain the necessary funds. The entrepreneur's project, the innovation, is a pair (S_e, X_e), where S_e contains the new states envisioned by the entrepreneur and $X_e : S_e \longrightarrow \mathbb{R}$ expresses the monetary return of the project as a function of the contingencies in S_e. At his end, the entrepreneur has (in his subjective opinion) a clear probabilistic view of the new world that he has envisioned. This is described by a probability measure P_e (we will be precise about the σ-algebra where this probability is defined, momentarily). At the other end, the financier, by facing a set of states he had never conceived of, perceives Ambiguity in the entrepreneur's description. This is described by the fact the financier evaluates the project by using a functional of the form (1), above. Two features place this problem outside the realm of standard contract theory. First, we have *heterogeneity* in the parties' beliefs: their views are different and, in fact, they are formed independently of each other. Second, one of the parties perceives *Ambiguity*, i.e. this party's beliefs are not represented by a probability measure. We are going to formalize this contracting problem in the remainder of this section and we will provide its solution in Section VII. In between, Section VI, we will discuss some related literature.

[2] In Schumpeter's work, the entrepreneur faces Ambiguity, while in our construction all of his uncertainty is reduced to Risk. This is not a substantial difference as we could allow for the entrepreneur to be described by non-additive criteria. This would result only in a technical complication without changing the essence of the problems we study.

A. Preliminaries

The scope of this subsection is to briefly discuss two aspects of the contracting problem that are seemingly technical. In fact, these aspects play a substantial role not only here but also elsewhere, for instance in the problem of whether or not a central authority is able to replicate the outcomes produced by an economy with innovation. In the present setting, the easiest way to grasp these aspects is also the most intuitive: just think of an entrepreneur and a financier coming together into a room; the former describes his project because he wants to get funding, the latter has to decide what to do.

The first issue has to do with the measurable structure on the set S_e. In our story, the financier is somebody who not only sees the innovation, i.e. the pair (S_e, X_e), for the first time in his life but has never conceived of it either. This implies that a contract between the financier and the entrepreneur may only be written on the basis of the information that is revealed in the room. The way to formalize this requirement is by endowing S_e with the coarsest σ-algebra which makes X_e measurable: this expresses precisely that *all the information available is derived from the description of the innovation*. We denote this the σ-algebra by Σ_e. Accordingly, the innovation can be written as $((S_e, \Sigma_e), X_e)$, and X_e is a random variable on (S_e, Σ_e). By Doob's measurability theorem (see [1, Theorem 4.41]), any measurable function g on (S_e, Σ_e) has the form $g = \zeta \circ X_e$, where ζ is a Borel-measurable function $\mathbb{R} \longrightarrow \mathbb{R}$. The Banach space of all bounded measurable functions on (S_e, Σ_e) (with $\|g\|_\infty = \sup_{s \in S} |g(s)|$) is denoted by $B(\Sigma_e)$ and the set of its positive elements by $B^+(\Sigma_e)$.

The second issue has to do with the probability P_e according to which the entrepreneur evaluates his own innovation. We assume that the entrepreneur declares truthfully this belief P_e, which is thus a common knowledge among the parties. Formally, this probability is just a mathematical representation of certain parts of the entrepreneur's project. Thus, de facto, we assume that the entrepreneur reveals truthfully some aspects of his project (precisely those that admit a representation in the form of a probabilistic assessment). We believe that this assumption sounds heavier than what it really is, and this is so for at least two reasons. First, when they come in contact with each other, the entrepreneur knows nothing about the financier (formally, this is encoded in the requirement on the σ-algebra). Thus, if he were to lie about those aspects of the project (i.e., declare a probability different from P_e), he would have no reason to think that this might increase his chances to get funded. Second, and perhaps more importantly, the financier's beliefs (in the non-additive sense) are formed independently of P_e. That is, the view the financier ends up with after being presented with the innovation would be the same whether P_e or any other probability is declared by the entrepreneur. Formally, what drives the feature that the financier might find the project worthwhile is not the probability P_e but the coefficient of ambiguity aversion $\alpha(X_e)$, which depends only on the random variable X_e and not on the probability P_e.

We have said that the probability P_e describes certain aspects of the entrepreneur's project. All the other aspects are encoded in the mapping X_e, which expresses the gains/losses that the project allegedly generates as a function of the new states. Needless to say, we do not make any assumption about how truthfully this part is revealed as this is the very essence of the contracting problem.

B. Definition of a contract

The formal definition of a contract is as follows.

Definition 3 *A contract between an entrepreneur and a financier is a pair* (H, Y)*, where* $H \geq 0$ *and* $Y \in B(\Sigma)$ *is such that* $Y \leq X_e$*.*

The interpretation is that a contract is a scheme according to which the financier pays H (which may be 0) to the entrepreneur and in exchange gets a claim on part of the amount $X_e(s)$, which obtains when $s \in S$ realizes. This claim may consist of all $X_e(s)$ or just a part of it. The amount that the entrepreneur gets when $s \in S$ realizes is denoted by $Y(s)$ (which may be 0). The definition includes as special cases the following types of contracts:

(a) The financier simply buys the project, and has no further obligation toward the entrepreneur. This obtain for $Y(s) = 0$, for every $s \in S$;

(b) The financiers acquires ownership of the project. When the state $s \in S$ realizes, he obtains the amount $X_e(s)$ and transfers $Y(s)$ to the entrepreneur;

(c) The entrepreneur retains ownership of the project, but commits to paying the amount $Z(s) = X_e(s) - Y(s)$ to the financier when $s \in S$ realizes. He does so in exchange for an up front (that is, before the uncertainty resolves) payment of H;

(d) The entrepreneur transfers part of the ownership to the financier in exchange for H, and the parties agree to a sharing rule that specifies that when $s \in S$ realizes the amount $Z(s) = X_e(s) - Y(s)$ goes to the financier and the amount $Y(s)$ goes to the entrepreneur.

In a static setting, the distinction between cases (b), (c) and case (d) is purely a matter of interpretation because the contract is formally the same. Differently, in case (a) one can actually talk of transfer of ownership. This is an important case, whose determination requires to characterize all those circumstances (as functions of the project X_e and of the parties' preferences) that lead to an optimal solutions with the feature that $Y(s) = 0$, for every $s \in S$. We plan on addressing this problem in a future inquiry. At the moment, we are going to be interested mainly in determining the form of a general contract, and in understanding the role played by Ambiguity in this type of problems.

Example 4 (Publishing) *In the case of "Author meets Publisher", the innovation is a new book, or music, or film, or other intellectual property. In publishing, the up-front payment H is called the "advance". The Publisher purchases the residual claim on the work, and contracts to pay the Author a royalty stream based on sales revenue, which corresponds to the function Y.*

Example 5 (Franchising) *While a franchising contract does not fall under the heading "contracting for innovation", it is worth noticing that such a contract has one of the forms above. In fact, in a franchising contract, a franchisee pays an initial lump-sum H to a franchiser who owns a certain franchise. In return, the franchisee receives the rights for a claim Y on a part of the revenue X of the franchise business.*

In the remainder of the paper, we are going to suppress the subscripts e since we are going to consider one entrepreneur only.

C. The entrepreneur

As previously mentioned, the entrepreneur has, in his subjective opinion, a clear probabilistic view of the new world S he has envisioned. This view is represented by a (countably additive) probability measure P on (S, Σ), which he uses to evaluate the possible contracts that he might sign. Formally,

Assumption 1e The entrepreneur evaluates contracts by means of the Subjective Expected Utility criterion

$$\int_S u_e(Y)dP, \qquad Y \in B(\Sigma)$$

where $u_e : \mathbb{R} \longrightarrow \mathbb{R}$ is the entrepreneur's utility for monetary outcomes.

Mainly for reasons of comparison with other parts of the contracting literature, we assume that the uncertainty on S is diffused. Precisely, we assume that

Assumption 2e $P \circ X^{-1}$ is nonatomic;

Finally, we make the following assumption on u_e:

Assumption 3e The entrepreneur's utility function u_e satisfies the following properties:

1. $u_e(0) = 0$;

2. u_e is strictly increasing and strictly concave;

3. u_e is continuously differentiable;

4. u_e is bounded.

Thus, in particular, we assume that the entrepreneur is risk averse.

D. The financier

When presented with innovation $((S, \Sigma), X)$, financier φ perceives Ambiguity. This is represented by the set $\mathcal{C}_\varphi$ (of probabilities on $SP \cup S_e$ whose conditional on SP is P_φ) which appears in (1), above. In order to describe the financier's evaluation of the innovation, we are going to restrict to a sub-class of the functionals of type (1): that of Choquet Expected Utility. This class still allows for a wide variety of behavior as these functionals need not be either concave or convex. In the Choquet Expected Utility model, introduced by Schmeidler [36], the functional (1) takes the form of an integral (in the sense of Choquet) with respect to a non-additive, monotone set function (a capacity). While the use of Choquet integrals has become quite common in the applications of decision theory, it is probably still not part of the toolbox of most professionals. Because of this, we have included a few basic facts about capacities and Choquet integrals in Appendix A. In sum, a financier φ is described as follows.

Assumption 1φ The financier's evaluates contract by means of the functional $\Phi : B(\Sigma) \longrightarrow \mathbb{R}$ defined by

$$\int_S u_\varphi(Z) dv, \qquad Z \in B(\Sigma)$$

where $u_\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ is the financier's utility for money, v is a capacity on Σ and the integral is taken in the sense of Choquet.

In line with Assumption 2e, we also assume

Assumption 2φ υ is a continuous capacity (see App. A)

Finally,

Assumption 3φ The financier is risk neutral. We take u_φ to be the identity on $\mathbb{R}$.

From now on, we are going to assume that the random variable X which describes the profitability of the project is a positive random variable, that is $X \in B^+(\Sigma)$. This is without loss of generality since it can always be obtained by suitably re-normalizing the parties utility functions.

E. The contracting problem

The problem of finding an optimal contract (H, Y) may be split into two parts: we first determine the optimal Y given H, and then use this to find the optimal H. In line with the description of economic agents of Section III, we have in mind situations characterized by two features: (a) the entrepreneur does not have initial wealth (at least to be devoted to the running the project); and, (b) while the entrepreneur is the sole potential provider of that innovation, there is competition among financiers to acquire it. Hence, the problem of finding an optimal contingent payment scheme Y can be formulated as follows

$$\sup_{Y \in B(\Sigma)} \int_S u_e(H - X(s) + Y(s))dP \tag{2}$$

$$\begin{aligned} s.t. \quad & 0 \leq Y \leq X \\ & \int_S (X - Y)d\upsilon \geq (1 + \rho)H \end{aligned}$$

The argument of the utility u_e in (2) is the entrepreneur's wealth as a function of the state $s \in S$ that will realize

$$W^e(s) = W_0^e + H - X(s) + Y(s)$$

where W_0^e denote the entrepreneur's initial wealth, which we set equal to zero. Clearly, $W^e(\cdot)$ is a measurable function on (S, Σ). The last constraint is the financier's *participation constraint*. It states that a necessary condition for the financier to offer the contract is that his evaluation of the random variable $X - Y$ (the amount that he receives, as a function of the state, if he signs the contract) be at least as high as the amount H that he has to pay up front to the entrepreneur. In fact, the financier's evaluation of $X - Y$ might have to be strictly higher than H since by funding the entrepreneur the financier might give up other investment opportunities, for instance those present in the standard asset market defined by SP, the publicly known states. This condition is expressed by the factor $(1 + \rho)$, where $\rho \geq 0$. The other constraint $(0 \leq Y \leq X)$ expresses two conditions: (a) the right-hand inequality states that, in each state of the world, the transfer from the financier to the entrepreneur does not exceed the profitability of the project; and, (b) the left-hand inequality states that if there is a transfer from the entrepreneur to the financier, this will not exceed the entrepreneur's initial wealth (which we have set equal to zero).

Once problem (2) is solved, the optimal H is determined by maximizing the financier's evaluation (given the optimal Y). While this is not the usual optimization problem as it involves maximizing a Choquet integral, its solution is known (e.g. [21, 23]) and it involves only a quantitative determination. Thus, we need to focus only on solving problem (2).

F. Truthful revelation of the profitability of the project

When studying a problem of contracting in a situation of uncertainty, one typically adds one more constraint to the ones we considered above. This is a monotonicity constraint that, in our case, would stipulate that the payment from the financier to the entrepreneur is an increasing function of X, that is $Y = \Xi \circ X$ for some increasing function $\Xi : \mathbb{R} \longrightarrow \mathbb{R}$. This would guarantee that the entrepreneur does not downplay the profitability of the project. For the moment, we are going to ignore this problem altogether. The reason is the following: in our main theorem, *we are going to show that the monotonicity of Y is a feature that appears in all optimal contracts that we determine*. Notice that this feature guarantees that, even in the case where the project profitability depends on (state-contingent) unobserved actions taken by the entrepreneur, there would be neither adverse selection nor moral hazard problems with our optimal contracts.

VI. RELATED LITERATURE

In our inquiry on the role of innovation, we have been led to studying a contracting problem where not only there is heterogeneity in the parties' beliefs but also the beliefs of one party are not additive as a reflection of the Ambiguity perceived by this party. The literature on contracting under heterogeneity and Ambiguity is not vast, but it does include notable contributions. We are going to focus only on the literature that directly relates to our work, and refer the reader to [25], [27] and [26] for other interesting issues (for instance, the effect of Ambiguity on the incompleteness of the contractual form [25]). Important contributions to the problem of existence and monotonicity of the optimal contract in situations of Ambiguity and/or heterogeneity have been made by [8], [9], [10], [14], [13] and [11]. Carlier and Dana [9] and [10] and Dana [14] show existence and monotonicity in settings characterized by the presence of Ambiguity but where there is no heterogeneity. Carlier and Dana [8] study a setting similar to ours, but impose the additional restriction that the capacity of one party is a distortion of the probability of the other party, thus retain a certain (weak) form of homogeneity. Chateauneuf, Dana and Tallon [13] allow for capacities (i.e., Ambiguity) on both sides, but they assume that both capacities are sub-modular distortions and that the state space is finite. Finally, Carlier and Dana [11] also allow for capacities on both sides, but demand that both capacities be distortions of the same measure, and that the heterogeneity be “small" (in a sense made precise in that paper). We contribute to this literature by proving an existence and monotonicity result in a setting where, while we have Ambiguity only on one side, we allow for any degree of heterogeneity. To this, we also add a characterization of the optimal contract that we obtain in Section VIII under the additional assumption of a supermodular capacity (not necessarily a probability distortion; in fact, our result is a bit more general than what is stated here; see Corollary 9 in Section VIII and Corollary 11 in Section IX).

VII. EXISTENCE AND MONOTONICITY OF THE OPTIMAL CONTRACT

In this section, we are going to show that the contracting problem (problem (2) of Section V) between the entrepreneur and the financier admits a solution. Moreover, we are going to show that this solution is increasing in X, thus clearing up the field from concerns of project's misrepresentation on the part of the entrepreneur. Our solution obtains under an assumption which guarantees a certain consistency between the financier's and the entrepreneur's assessments of the uncertainty. The formal property is stated in the following definition, which extends to a setting with Ambiguity a concept originally introduced in Ghossoub [17].

Definition 6 *Let v be a capacity on Σ, P a measure on the same σ-algebra and let X be a random variable on (S,Σ). We say that v is (P,X)-vigilant if for any Y_1,$Y_2 \in B^+(\Sigma)$ such that*

(i) Y_1 and Y_2 have the same distribution under P; and

(ii) Y_2 *and* X *are comonotonic*[3]*,*

the following holds

$$\int (X - Y_2) dv \geq \int (X - Y_1) dv$$

Loosely, to say that v *is* (P, X)-vigilant means that the financier considers the entrepreneur's description (P, X) of the project sufficiently credible. Note that this is a subjective statement on the part of the financier. In fact, one can depict the following story. An entrepreneur envisions the new world S and comes up with his new idea (P, X). Then, he goes to a financier to ask for funding, and tells him about the new world S and the project (P, X). The financier forms his view of S, which is described by v, and decides how credible the entrepreneur's project is. If he deems it sufficiently credible, then they would start negotiating. If not, the entrepreneur would take leave and seek a financier with a different opinion. Thus, the appearance of assumptions of the vigilance-type should not be surprising, as ultimately these are conditions for both parties to believe in the mutual profitability of the project. Before proceeding, we would like to stress that, in the special case where the capacity v is a measure, the assumption of vigilance is a weakening of the *monotone likelihood ratio property* frequently assumed in the contracting literature to deal with problems stemming from the asymmetry in the information. We refer the reader to Ghossoub [18] for the relation between the two properties in a context of Risk. We can now state our main result.

Theorem 7 *If* v *is* (P, X)*-vigilant, then Problem (2) admits a solution* Y *which is comonotonic with* X*.*

The proof of the Theorem is in Appendix B.

VIII. AMBIGUITY-LOVING FINANCIER AND A CHARACTERIZATION OF THE OPTIMAL CONTRACT

In Section III, we said that a fairly general description of the way financiers deal with Ambiguity would be that provided by the functionals of the form (see (1), Section III)

$$\Phi(X_e) = \alpha(X_e) \min_{Q \in \mathcal{C}_\varphi} \int u_\varphi(X_e) dQ + (1 - \alpha(X_e)) \max_{Q \in \mathcal{C}_\varphi} \int u_\varphi(X_e) dQ$$

[3] For the definition of comonotonic functions, see Appendix A.

where the coefficient $\alpha(\cdot)$ is allowed to vary with the project to be evaluated. The variability of the coefficient expresses the financier's preference for certain projects over others, maybe because they are closer to his subjective view (we pointed out in Section III that we can allow for financiers to have a subjective view by simply re-defining the function $\alpha(\cdot)$). A natural special case of this description obtains when the coefficient $\alpha(\cdot)$ is constant and equal to 0. This would represent the case where the financier is not really concerned about the kind of Ambiguity he faces, but cares only about the presence of Ambiguity, and he is willing to bet on its resolution. Thus, the financier's functional is given by

$$\Phi(X_e) = \max_{Q \in \mathcal{C}_\varphi} \int u_\varphi(X_e)\, dQ \tag{3}$$

By a theorem of Schmeidler [35], a subclass of these functionals obtains as a special case of Choquet integrals. Precisely, a Choquet integral can be written in the form (3) if and only if the capacity that defines it is supermodular (see Appendix A). In this case, we can give a characterization of the solution whose existence we proved in Theorem 7. Proposition 8 below shows that, when the capacity representing the financier is supermodular, the optimal solution to the contracting problem (2) is the same as the solution of another contracting problem, which involves heterogeneity but not Ambiguity. It is important to stress, as the proof of Proposition 8 makes it clear, that this is not a statement about the type of uncertainty involved in this problem (2) but only a devise which allows us to characterize the solution. The usefulness of the equivalence proved in Proposition 8 stems from the fact that the solution can now be characterized by using the methods introduced in Ghossoub [17]. In fact, under some mild additional conditions, this solution can even be characterized analytically (see Ghossoub [18]).

So, let us assume that the capacity υ representing the financier in **Assumption 1**φ is supermodular. Then, the functional Φ takes the form (3). The set $\mathcal{C}_\varphi$ is called the *anti-core* of υ (and is non-empty). For $Q \in \mathcal{C}_\varphi$, consider the following problem

$$\sup_{Y \in B(\Sigma)} \int_S u_e(H - X(s) + Y(s))\, dP \tag{4}$$

$$\begin{aligned} s.t. \quad & 0 \leq Y \leq X \\ & \int_S (X - Y)\, dQ \geq (1+\rho)H \end{aligned}$$

That is, problem (4) is contracting problem like (2) but (ideally) involves a financier that is an

Expected Utility maximizer, with $Q \in \mathcal{C}_\varphi$ being the probability representing the financier. Let us denote by $Y^*(Q)$ the optimal solution of problem (4) for $Q \in \mathcal{C}_\varphi$.

Proposition 8 *If the capacity υ in* ***Assumption 1****φ is supermodular and (P, X)-vigilant, then there exists a $Q^* \in anticore(\upsilon)$ such that $Y^*(Q^*)$ solves the contracting problem (2).*

Inspection of the proof of Proposition 8 (Appendix B) shows that this result can be extended to general functionals of the form (3), that is functionals of the form (3) that are not necessarily Choquet integrals.

Corollary 9 *Assume that in problem (2) the financier is described by a functional of the form*

$$\Phi(X_e) = \max_{Q \in \mathcal{C}} \int u_\varphi(X_e) dQ$$

where is a set $\mathcal{C}$ of probability measures on (S, Σ). If every $Q \in \mathcal{C}$ is (P, X)-vigilant, then then there exists a $Q^* \in \mathcal{C}$ such that $Y^*(Q^*)$ solves the contracting problem.

In the next section, where we discuss insurance contracts, we will give a pictorial description of this type of solutions.

IX. INSURANCE CONTRACTS

In an insurance framework, one party (the insured) pays a premium in return for a (state-contingent) indemnity provided by the other party (the insurer). This problem has been studied by Arrow [4], Borch [7] and Raviv [31] under the assumptions that (1) both parties are Expected Utility maximizers (there is no Ambiguity); (2) both parties entertain the same beliefs (there is no heterogeneity); and, (3) the insured is risk-averse and the insurer is risk-neutral. The solution that they provided shows that the optimal contract takes the form of a *deductible*.

This now classical result was extended only recently to the case of heterogeneity in the parties' beliefs (but with no Ambiguity) by Ghossoub [18]. Insurance problems with Ambiguity have been studied in the papers we mentioned in Section VI.

By making the natural assumption that the insured is an Expected Utility maximizer while the insurer might perceive some Ambiguity, the problem of optimal insurance takes the following form:

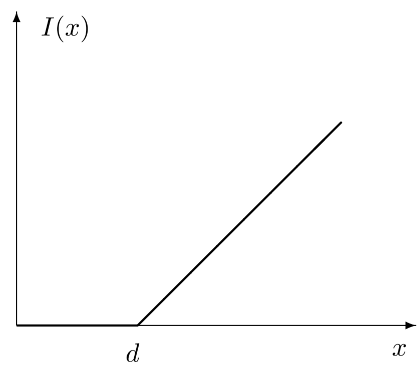

FIG. 1: A deductible contract.

$$\sup_{Y \in B(\Sigma)} \int_S u_i(W_0 + H - X(s) + Y(s))dP \tag{5}$$
$$s.t. \quad 0 \leq Y \leq X$$
$$\int_S -Y dv \geq H' = (1+\rho)H$$

In problem (5), i is the insured; u_i is his utility and the argument of u_i is the wealth of the insured as a function of the state (W_0 is the insured's initial wealth); X is the insurable loss and Y is the indemnity; finally, H is the negative of the premium Π, that is, $H = -\Pi$, and ρ is a *loading* on the premium. The last constraint ($\int -Y dv \geq H'$) is the insurer's participation constraint, where $\int \cdot dv$ is the Choquet integral describing how the insurer deals with the Ambiguity that he perceives. In the special case of a supermodular v, this problem becomes

$$\sup_{Y \in B(\Sigma)} \int_S u_i(W_0 + H - X(s) + Y(s))dP$$
$$s.t. \quad 0 \leq Y \leq X$$
$$\min_{Q \in \mathcal{C}} \int_S Y dv \leq (1+\rho)\Pi$$

where $\mathcal{C} = anticore(v)$. Just like we did in the previous section, we can consider a family of contracting problems parametrized by the set $\mathcal{C}$. Each problem in this family is of the form (5) with the only difference that the insurer's Choquet integral is replaced by the Lebesgue integral

$\int \cdot dQ$, $Q \in \mathcal{C}$. We denote by $Y^*(Q)$ the solution of this problem. A simple adaptation of the proof of Proposition 8 then shows that

Corollary 10 *If v is supermodular and (P, X)-vigilant, then there exists a $Q^* \in \mathcal{C}$ such that $Y^*(Q^*)$ solves the insurance problem (5).*

In the same vein as Corollary 9, Section VIII, we also have

Corollary 11 *Assume that in the insurance problem the insurer is described by a functional of the form*

$$\mathcal{I}(Y) = \min_{Q \in \mathcal{C}} \int Y dv$$

where $\mathcal{C}$ is a set of probability measures on (S, Σ). If every $Q \in \mathcal{C}$ is (P, X)-vigilant, then there exists a $Q^* \in \mathcal{C}$ such that $Y^*(Q^*)$ solves the insurance problem.

In the cases covered by Corollary 11 (which includes Corollary 10), the characterization of the optimal contract then follows from the result of Ghossoub [18].

Corollary 12 *The optimal contract $Y^*(Q^*)$ in Corollary 11 is a generalized deductible.*

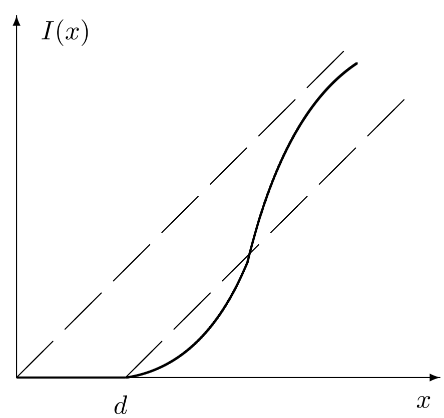

FIG. 2: An example of a generalized deductible contract.

Thus, the difference with respect to the no-heterogeneity and no-Ambiguity setting of Arrow-Borch-Raviv consists of the non-linearity of the risk-sharing schedule. The source of this difference is clear. The Arrow-Borch-Raviv is a pure risk-sharing result: the two parties sign the contract because of the different shapes of the utility functions (one is risk-averse, the other is risk-neutral), but they have exactly the same view of the uncertainty. When, as in our setting, the parties differ also because of their views about uncertainty, intuitively they have to share uncertainty in addition to risk. By taking the Arrow-Borch-Raviv case as a reference point, we could interpret the concave parts of the optimal schedule as an indication that the insured is more optimistic about certain outcomes than the insurer, with the situation being reversed in the convex parts.

Unlike Corollary 9 that has the re-formulation given by Corollary 11 in the insurance framework, a re-formulation of Theorem 7 is not straightforward. Inspection of the proof of Theorem 7 shows that the main difficulty in transferring that result to an insurance framework resides in the lack of homogeneity of the Choquet integral (that is, for Choquet integrals in general $\int -Y dv \neq -\int Y dv$). This difficulty can be circumvented by replacing the Choquet integral with the Šipoš integral (or symmetric Choquet integral; see Appendix A). Unlike the Choquet integral, the Šipoš integral is homogeneous, and the proof of Theorem 7 goes through in the insurance setting as well. We thus have:

Corollary 13 *Assume that in the insurance problem the insurer is described by a Šipoš integral and that v is (P, X)-vigilant. Then, the insurance problem admits a solution Y which is comonotonic with X.*

APPENDIX A. Background material

A.1 The Choquet integral

Here, we summarize the basic definitions about capacities, Choquet integrals and Šipoš integrals. The proofs of the statements listed below can be found, for instance, in [24].

Definition 14 *A (normalized) capacity on a measurable space* (S, Σ) *is a set function* $\upsilon : \Sigma \longrightarrow [0,1]$ *such that*

(1) $\upsilon(\varnothing) = 0$;

(2) $\upsilon(S) = 1$; *and*

(3) $A, B \in \Sigma$ *and* $A \subset B \Longrightarrow \upsilon(A) \leq \upsilon(B)$.

Definition 15 *A capacity* υ *on* (S, Σ) *is continuous from above (resp. below) if for any sequence* $\{A_n\} \subset \Sigma$ *such that* $A_{n+1} \subset A_n$ *(resp.* $A_{n+1} \supset A_n$*) for each* n*, it holds that*

$$\lim_{n\to\infty} \upsilon(A_n) = \upsilon\left(\bigcap_{n=1}^{\infty} A_n\right) \qquad (resp.\ \lim_{n\to\infty} \upsilon(A_n) = \upsilon\left(\bigcup_{n=1}^{\infty} A_n\right))$$

A capacity that is continuous both from above and below is said to be continuous.

Definition 16 *Given a capacity* υ *and a function* $\psi \in B(\Sigma)$*, the Choquet integral of* ψ *w.r.t.* υ *is defined by*

$$\int \psi d\upsilon = \int_0^{\infty} \upsilon(\{s \in S : \psi(s) \geq t\})dt + \int_{-\infty}^{0} [\upsilon(\{s \in S : \psi(s) \geq t\}) - 1]dt$$

where the integrals on the RHS are taken in the sense of Riemann.

Unlike the Lebesgue integral, the Choquet integral is not additive. One of its characterizing properties, however, is that it respects additivity on comonotonic functions.

Definition 17 *Two functions* $Y_1, Y_2 \in B(\Sigma)$ *are comonotonic if for all* $s, s' \in S$

$$[Y_1(s) - Y_1(s')][Y_2(s) - Y_2(s')] \geq 0$$

As noticed, if $Y_1, Y_2 \in B(\Sigma)$ are comonotonic then

$$\int (Y_1 + Y_2)d\upsilon = \int Y_1 d\upsilon + \int Y_2 d\upsilon$$

Definition 18 *A capacity υ on (S, Σ) is submodular if for any $A, B \in \Sigma$*

$$\upsilon(A \cup B) + \upsilon(A \cap B) \leq \upsilon(A) + \upsilon(B)$$

It is supermodular is the reverse inequality holds for any $A, B \in \Sigma$.

A Choquet integral $\int \cdot d\upsilon$ is concave (resp. convex) iff υ is submodular (resp. supermodular).

The *Šipoš integral*, or the *symmetric Choquet integral* (see [28]), is a functional $\check{S} : B(\Sigma) \longrightarrow \mathbb{R}$ defined by

$$\check{S}(Y) = \int Y^+ d\upsilon - \int Y^- d\upsilon$$

where the integrals on the RHS are taken in the sense of Choquet and Y^+ (resp. Y^-) denotes the positive (resp. negative) part of $Y \in B(\Sigma)$. Obviously, the Šipoš integral coincides with the Choquet integral for positive functions.

A.2 Nondecreasing rearrangements

The results in this Appendix are taken from Ghossoub [18] to which we refer the reader for proofs and additional results.

A.2.1 The Nondecreasing Rearrangement

Let $(S, \mathcal{G}, P)$ be a probability space, and let $X \in B^+(\mathcal{G})$ be a continuous random variable (i.e., $P \circ X^{-1}$ is non-atomic) with range $X(S) = [0, M]$. Let Σ be the σ-algebra generated by X, and let

$$\phi(B) = P(\{s \in S : X(s) \in B\}) = P \circ X^{-1}(B)$$

for any Borel subset B of $\mathbb{R}$.

Let $I : [0, M] \rightarrow [0, M]$ be any Borel-measurable map. Then there exists a ϕ-a.s. unique nondecreasing Borel-measurable map $\tilde{I} : [0, M] \rightarrow [0, M]$ which is ϕ-equimeasurable with I, in the sense that for any $\alpha \in [0, M]$,

$$\phi(\{t \in [0, M] : I(t) \leq \alpha) = \phi(\{t \in [0, M] : \tilde{I}(t) \leq \alpha)$$

$\tilde{I}$ is called the *nondecreasing ϕ-rearrangement of I*. Now, define $Y = I \circ X$ and $\tilde{Y} = \tilde{I} \circ X$. Since both I and $\tilde{I}$ are Borel-measurable mapping of $[0, M]$ into itself, it follows that $Y, \tilde{Y} \in B^+(\Sigma)$. Note also that $\tilde{Y}$ is non-decreasing in X, in the sense that if $s_1, s_2 \in S$ are such that $X(s_1) \leq X(s_2)$ then $\tilde{Y}(s_1) \leq \tilde{Y}(s_2)$ and that Y and $\tilde{Y}$ are equimeasurable, that is for any $\alpha \in [0, M]$, $P(\{s \in S : Y(s) \leq \alpha) = P(\{s \in S : \tilde{Y}(s) \leq \alpha)$.

We will call $\tilde{Y}$ a *nondecreasing* P*-rearrangement of* Y *with respect to* X, and we shall denote it by $\tilde{Y}_P$. Note that $\tilde{Y}_P$ is P-a.s. unique. Note also that if Y_1 and Y_2 are P-equimeasurable and if $Y_1 \in \mathcal{L}_1(S, \mathcal{G}, P)$, then $Y_2 \in \mathcal{L}_1(S, \mathcal{G}, P)$ and $\int \psi(Y_1)dP = \int \psi(Y_2)dP$ for any measurable function ψ such that the integral exists.

A.2.2 Supermodularity and Hardy-Littlewood Inequalities

A *partially ordered set* (*poset*) is a pair $(A, \succsim)$ where $\succsim$ is a reflexive, transitive and antisymmetric binary relation on A. For any $x, y \in A$, we denote by $x \vee y$ (resp. $x \wedge y$) the least upper bound (resp. greatest lower bound) of the set $\{x, y\}$. A poset $(A, \succsim)$ is a *lattice* when $x \vee y, x \wedge y \in A$ for every $x, y \in A$. For instance, the Euclidian space $\mathbb{R}^n$ is a lattice for the partial order $\succcurlyeq$ defined as follows: for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $y = (y_1, ...y_n) \in \mathbb{R}^n$, we write $x \succcurlyeq y$ when $x_i \geq y_i$, for each $i = 1, \ldots, n$.

Definition 19 *Let $(A, \succsim)$ be a lattice. A function $L : A \longrightarrow \mathbb{R}$ is supermodular if for each $x, y \in A$*

$$L(x \vee y) + L(x \wedge y) \geq L(x) + L(y)$$

In particular, a function $L : \mathbb{R}^2 \to \mathbb{R}$ is *supermodular* if for any $x_1, x_2, y_1, y_2 \in \mathbb{R}$ with $x_1 \leq x_2$ and $y_1 \leq y_2$, we have

$$L(x_2, y_2) + L(x_1, y_1) \geq L(x_1, y_2) + L(x_2, y_1)$$

It is easily seen that the supermodularity of a function $L : \mathbb{R}^2 \to \mathbb{R}$ is equivalent to the function $\eta(y) = L(x + h, y) - L(x, y)$ being nondecreasing for any $x \in \mathbb{R}$ and $h \geq 0$.

Example 20 *The following are useful examples of supermodular functions:*

(1) If $g : \mathbb{R} \to \mathbb{R}$ is concave, and $a \in \mathbb{R}$, then the function $L_1 : \mathbb{R}^2 \to \mathbb{R}$ defined by $L_1(x, y) = g(a - x + y)$ is supermodular.

(2)The function $L_3 : \mathbb{R}^2 \to \mathbb{R}$ defined by $L_3(x, y) = -(y - x)^+$ is supermodular.

Lemma 21 *Let $Y \in B^+(\Sigma)$ and let $\tilde{Y}_P$ the nondecreasing P-rearrangement of Y with respect to X. Then,* *(a)* $0 \le Y \le X$ *implies* $0 \le \tilde{Y}_P \le X$*;*

(b) If L *is a supermodular,* $P \circ X^{-1}$*-integrable function on the range of* X*, then*

$$\int L(X,Y)dP \le \int L(X,\tilde{Y}_P)dP$$

APPENDIX B.

B.1 Proof of Theorem 7

Let us denote by $\mathcal{F}_{SB}$ the feasibility set for Problem 2

$$\mathcal{F}_{SB} = \left\{ Y \in B(\Sigma) : 0 \le Y \le X \text{ and } \int (X - Y)dv \ge (1+\rho)H = H' \right\}$$

and let $\mathcal{F}_{SB}^{\uparrow}$ be the set of all the $Y \in \mathcal{F}_{SB}$ which, in addition, are comonotonic with X:

$$\mathcal{F}_{SB}^{\uparrow} = \{Y = I \circ X \in \mathcal{F}_{SB} : I \text{ is nondecreasing}\}$$

Lemma 22 *If* v *is* (P,X)*-vigilant, then for each* $Y \in \mathcal{F}_{SB}$ *there exists a* $\tilde{Y} \in \mathcal{F}_{SB}$ *such that*

(1) $\tilde{Y}$ *is comonotonic with* X

(2) $\int u_e(H - X + \tilde{Y})dP \ge \int u_e(H - X + Y)dP$

(3) $\int (X - \tilde{Y})dv \ge \int (X - Y)dv$

Proof. Choose any $Y = I \circ X \in \mathcal{F}_{SB}$, and let $\tilde{Y}_P$ denote the nondecreasing P-rearrangement of Y with respect to X. Then (i) $\tilde{Y}_P = \tilde{I} \circ X$ where $\tilde{I}$ is nondecreasing, and (ii) $0 \le \tilde{Y}_P \le X$, by Lemma 21. Furthermore, since v is (P,X)-vigilant, it follows that $\int (X - \tilde{Y}_P)dv \ge \int (X - Y)dv$. But $\int (X - Y)dv \ge H'$ since $Y \in \mathcal{F}_{SB}$. Hence, $\tilde{Y}_P \in \mathcal{F}_{SB}^{\uparrow}$. It remains to show (2). Since the utility u_e is concave (**Assumption 3**e), the function $\mathcal{U}(x,y) = u_e(H - x + y)$ is supermodular (see App. A). Then, by Lemma 21

$$\int u_e(H - X + \tilde{Y})dP \ge \int u_e(H - X + Y)dP$$

■

Proof of Theorem 7. By Lemma 22, we can choose a maximizing sequence $\{Y_n\}_n$ in $\mathcal{F}_{SB}^{\uparrow}$ for Problem 2. That is,

$$\lim_{n\to+\infty}\int u_e(H-X+Y_n)dP = N \equiv \sup_{Y\in B^+(\Sigma)}\{u_e(H-X+Y_n)dP\} < +\infty$$

Since $0 \le Y_n \le X \le M \equiv \|X\|_\infty$, the sequence $\{Y_n\}_n$ is uniformly bounded. Moreover, for each $n \ge 1$ we have $Y_n = I_n \circ X$, with $I_n : [0, M] \to [0, M]$. Consequently, the sequence $\{I_n\}_n$ is a uniformly bounded sequence of nondecreasing Borel-measurable functions. Thus, by Helly's First Theorem [12, Lemma 13.15] (aka Helly's Compactness Theorem), there is a nondecreasing function $I^* : [0, M] \to [0, M]$ and a subsequence $\{I_m\}_m$ of $\{I_n\}_n$ such that $\{I_m\}_m$ converges pointwise on $[0, M]$ to I^*. Hence, I^* is also Borel-measurable, and so $Y^* = I^* \circ X \in B^+(\Sigma)$ is such that $0 \le Y^* \le X$. Moreover, the sequence $\{Y_m\}_m$, $Y_m = I_m \circ X$, converges pointwise to Y^*. Thus, the sequence $\{X - Y_m\}_m$ is uniformly bounded and converges pointwise to $\{X - Y^*\}$. By the Assumption that υ is continuous (**Assumption 2**φ), it follows from a Dominated Convergence-type Theorem [28, Theorem 7.16][4] that

$$H' \le \lim_{m\to+\infty}\int (X - Y_m)d\nu = \int (X - Y^*)d\nu$$

and so $Y^* \in \mathcal{F}_{SB}^{\uparrow}$. Now, by continuity and boundedness of the function u_e, and by Lebesgue's Dominated Convergence Theorem [1, Theorem 11.21], we have

$$\begin{aligned}\int u_e(H-X+Y^*)dP &= \lim_{m\to+\infty}\int u_e(H-X+Y_m)dP \\ &= \lim_{n\to+\infty}\int u_e(H-X+Y_n)dP = N\end{aligned}$$

Hence Y^* solves Problem 2. ■

B.2 Proof of Proposition 8

Proof. Let Y^{**} denote a solution of the contracting problem (2). Fix $Q \in \mathcal{C}_\varphi$, and let $Y^*(Q)$ be the optimal solution of problem (4) for $Q \in \mathcal{C}_\varphi$. Then, $Y^*(Q)$ satisfies

$$0 \le Y^*(Q) \le X$$

$$\int_S (X - Y^*(Q))dQ \ge (1+\rho)H$$

[4] The theorem of Pap [28] is for the *Šipoš integral*, or the *symmetric Choquet integral*. However, the latter coincides with the Choquet integral for nonnegative functions, see App. A.

Hence,

$$\max_{R\in\mathcal{C}_\varphi}\int_S (X-Y^*(Q))dR \geq \int_S (X-Y^*(Q))dQ \geq (1+\rho)H$$

which shows that $Y^*(Q)$ is feasible in problem (2). Since Y^{**} solves problem (2), we must have

$$\int_S u_e(H-X+Y^{**})dP \geq \int_S u_e(H-X+Y^*(Q))dP \tag{6}$$

and to conclude the proof, it suffices to find some $Q^{**}\in\mathcal{C}_\varphi$ such that inequality (6) holds as an equality. Suppose, by the way of contradiction, that no such a Q^{**} exists. Then, for all $Q\in\mathcal{C}_\varphi$ it holds that

$$\int_S u_e(H-X+Y^{**})dP > \int_S u_e(H-X+Y^*(Q))dP \tag{7}$$

Since, by definition, $Y^*(Q)$ solves the problem of type (4) defined by Q, inequality (7) implies Y^{**} must not be feasible in any the problems of type (4), that is for all $Q\in\mathcal{C}_\varphi$

$$\int_S (X-Y^{**})dQ < (1+\rho)H$$

But, by the feasibility of Y^{**} in problem (2), we have that for all $Q\in\mathcal{C}_\varphi$

$$\int_S (X-Y^{**})dQ < (1+\rho)H \leq \max_{R\in\mathcal{C}_\varphi}\int_S (X-Y^{**})dR$$

which, since $(X-Y^*)\in B(\Sigma)$, contradicts the fact that $\mathcal{C}_\varphi$ is compact and convex. ■

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[1] C.D. Aliprantis and K.C. Border, *Infinite Dimensional Analysis - 3^{rd} edition*, Springer-Verlag, 2006.

[2] M. Amarante and E.S. Phelps, *The entrepreneurial economy*, Kauffman Foundation report, 2005.

[3] M. Amarante, O. Ozgur and E.S. Phelps, *The entrepreneurial economy II: equilibria and welfare*, mimeo 2011.

[4] K.J. Arrow, *Essays in the Theory of Risk-Bearing*, Chicago: Markham Publishing Company, 1971.

[5] R. Bojilov and E.S. Phelps, *Dynamism and Economic Performance: A preliminary report*, The Center of Capitalism and Society WP 60, 2010.

[6] M. Boldrin and D.K. Levine, *Perfectly competitive innovation*, Journal of Monetary Economics, 55(3), 435-453, 2008.

[7] K.H. Borch, *The Mathematical Theory of Insurance: An Annotated Selection of Papers on Insurance Published 1960-1972*, Lexington Books, 1974.

[8] G. Carlier and R.A. Dana, *Core of convex distortions of a probability*, Journal of Economic Theory, 113(2), 199-222, 2003.

[9] G. Carlier and R.A. Dana, *Pareto efficient insurance contract when the insurer's cost function is discontinuous*, Economic Theory, 21(4) 871-893, 2003.

[10] G. Carlier and R.A. Dana, *Rearrangements inequality in non-convex insurance models*, Journal of Mathematical Economics, 41(4-5) 483-503, 2005.

[11] G. Carlier and R.A. Dana, *Two-person efficient risk-sharing and equilibria for concave law-invariant utilities*, Economic Theory 36(2) 189-223, 2008.

[12] N.L. Carothers, *Real Analysis*, Cambridge University Press, 2000.

[13] A. Chateauneuf, R.A. Dana and J-M. Tallon, *Optimal risk-sharing rules and equilibriam with Choquet-expected-utility*, Journal of Mathematical Economics 34(20 191-214, 2000.

[14] R.A. Dana, *On equilibria when agents have multiple priors*, Annals of Operation Research 114(1) 105-115, 2002.

[15] P. Ghirardato, F. Maccheroni, and M. Marinacci, *Differentiating Ambiguity and Ambiguity Attitude*, Journal of Economic Theory, **118** (2004), no. 2, 133–173.

[16] P. Ghirardato and M. Siniscalchi, *A More Robust Definition of Multiple Priors*, mimeo (2010).

[17] M. Ghossoub, *Contracting under Heterogeneous Beliefs*, Ph.D. Thesis, University of Waterloo. Dept of Statistics and Actuarial Science, 2011.

[18] M. Ghossoub, *On the demand for contingent claims under heterogeneous uncertainty*, mimeo 2011.

[19] I. Gilboa and M. Marinacci, *Ambiguity and the Bayesian Paradigm*, IGER WP 379, 2011.

[20] I. Gilboa and D. Schmeidler, *Maxmin Expected Utility with a Non-Unique Prior*, Journal of Mathematical Economics, **18** (1989), no. 2, 141–153.

[21] X.D. He and X.Y. Zhou, *Portfolio Choice via Quantiles*, Mathematical Finance, **21** (2011), no. 2, 203–231.

[22] H.T. Hoon and E.S. Phelps, *Innovation and Employment*, The Center of Capitalism and Society WP 61, 2011.

[23] H. Jin and X.Y. Zhou, *Behavioral Portfolio Selection in Continous Time*, Mathematical Finance, **18** (2008), no. 3, 385–426.

[24] M. Marinacci and L. Montrucchio, *Introduction to the Mathematics of Ambiguity*, in Uncertainty in Economic Theory: Essays in honor of David Schmeidler's 65th birthday, I. Gilboa (ed.), Routledge 2004.

[25] S. Mukerji, *Ambiguity aversion and the incompleteness of the contractual form*, The American Economic Review, 88(5) 1207-1232, 1998.

[26] S. Mukerji, *A survey of some applications of the idea of ambiguity aversion in economics*, International Journal of Approximate Reasoning, 24(2-3) 221-234, 2000.

[27] S. Mukerji and J-M. Tallon, *An overview of economic applications of David Schmeidler's models of decision making under uncertainty*, in Uncertainty in Economic Theory: Essays in honor of David

Schmeidler's 65th birthday, I. Gilboa (ed.), Routledge 2004.

[28] E. Pap, *Null-Additive Set Functions*, Kluwer Academic Publishers, 1995.

[29] E.S. Phelps, *Toward a Model of Innovation and Performance Along the Lines of Knight, Keynes, Hayek and M. Polanyí*, The Center of Capitalism and Society WP 19, 2006.

[30] E.S. Phelps, *The Economic Prosperity of Nations: Prosperity Depends on Dynamism, Dynamism on Institutions*, The Center of Capitalism and Society 2, 2005.

[31] A. Raviv, *The Design of an Optimal Insurance Policy*, American Economic Review, **69** (1979), no. 1, 84–96.

[32] J. Reinganum, *The timing of innovation: Research, development, and diffusion*, Handbook of Industrial Organization, R. Schmalensee & R. Willig (ed.), vol. 1, ch. 14, 849-908, Elsevier 1989.

[33] P. Roemer, *Increasing Returns and Long-run Growth*, Journal of Political Economy, 94(5),1002-37, 1986.

[34] Primarily Innovation and IP Oriented, http://rufuspollock.org/economics/biblio/

[35] D. Schmeidler, *Integral Representation without Additivity*, Proceedings of the AMS **97**, 255-61, 1986.

[36] D. Schmeidler, *Subjective Probability and Expected Utility without Additivity*, Econometrica, **57** 571–587,1989.

[37] J. Schumpeter, Théorie de l'évolution économique, Dalloz, 1935.

[38] S. Scotchmer, Innovation and Incentives, The MIT Press, 2004.